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Compact Operators on Matrix Domains via the Hausdorff Measure of Non-Compactness

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Abstract **Original Research Article**

This paper gives a quantitative compactness framework for operators on matrix domains of triangular matrices. Let XA={x:Ax∈X} be equipped with the natural norm induced by an invertible triangle A. Because UAx=Ax is an isometric isomorphism, the Hausdorff measure of non-compactness of an operator defined on XA can be evaluated after transferring the operator to the model space X. For a matrix B:XA→Y, the associated operator is represented by B̂=BA-1; for endomorphisms of XA the relevant conjugate is ABA-1. Exact formulas are obtained for operators from ℓp(A) into c_0, and two-sided estimates are obtained for maps into c. For targets ℓr, 1≤r<∞, compactness is characterized by vanishing tail operator norms. Specializations to ℓ1(A) give an explicit column-tail criterion, while a Hilbert-Schmidt condition yields a practical sufficient condition on ℓ2(A). Examples based on first-order weighted differences show how the abstract transfer becomes a computable condition on matrix coefficients. The results extend the standard use of the Hausdorff measure of non-compactness on BK spaces to a general matrix-domain setting and clarify the role of the inverse triangle in compactness tests.

Keywords: Hausdorff measure of non-compactness; compact operator; matrix domain; BK space; infinite matrix; difference sequence space.

1. Introduction

Compactness of infinite matrix operators is a central question in the theory of sequence spaces because boundedness alone does not control the behavior of tails. The Hausdorff measure of non-compactness provides a quantitative way to detect the failure of compactness: a bounded operator is compact precisely when the measure of non-compactness of the image of the unit ball is zero. This method has been applied extensively to BK spaces, generalized means, difference spaces and matrix domains [1–7]. Foundational work on difference and matrix-domain sequence spaces includes Kızmaz [10], Kirişçi and Başar [9], and Malkowsky and Rakočević [11]. Compactness results for difference and matrix-domain settings were further developed by Maji, Manna and Srivastava [8], Mursaleen and Noman [12], and Djolović and Malkowsky [13].

Matrix domains introduce an additional layer. If A is a triangle and X_A={x:Ax∈X}, then a matrix B acting on X_A is not most naturally analyzed by its original rows b_n. The correct rows for operator estimates are those of the associated matrix $\hat{B}=BA^{-1}$, because $x=A^{-1}y$ with y∈X. This observation is standard in matrix-transformation theory, but the Hausdorff measure makes the reduction especially powerful: when X_A carries the natural norm $\|x\|_A=\|Ax\|_X$, the map $U_Ax=Ax$ is an isometry, so bounded sets and their measures of non-compactness are transported without distortion.

The aim of this paper is to formulate that transfer principle at the operator level and then derive explicit compactness criteria for common target spaces. The strongest formulas occur when the target has a canonical Schauder basis whose tail projections compute the Hausdorff measure exactly. In c_0 and ℓ_r, $1\le r<\infty$, the measure of a bounded set is determined by the asymptotic size of its coordinate tails. For c, a comparable two-sided estimate holds after subtracting the limit functional. These facts lead directly to row-norm and tail-matrix criteria for operators from $\ell_p(A)$.

The paper is organized as follows. Section 2 recalls the Hausdorff measure of non-compactness and the projection formulas used later. Section 3 proves invariance under the isometric matrix-domain transform. Section 4 gives exact criteria for maps from $\ell_p(A)$ into c_0 and compactness tests for maps into c. Section 5 treats ℓ_r targets and obtains an explicit column-tail condition for $\ell_1(A)$. Section 6 discusses endomorphisms and compact perturbations. Section 7 works out examples for weighted first differences and Hilbert-Schmidt kernels.

2. Hausdorff measure of non-compactness

Let E be a Banach space and let $Q\subset E$ be bounded. The Hausdorff (ball) measure of non-compactness is

$$\chi_E(Q)=\inf\{\varepsilon>0 : Q \text{ has a finite } \varepsilon\text{-net in } E\}. \quad (2.1)$$

Thus $\chi_E(Q)=0$ if and only if Q is relatively compact. For $T\in B(E,F)$, define the operator measure of non-compactness by

$$\|T\|_\chi = \chi_F(T(B_E)), \qquad (2.2)$$

where B_E is the closed unit ball. Then T is compact if and only if $\|T\|_\chi=0$. These facts underlie the compactness criteria developed by Mursaleen, Noman, Malkowsky and others for matrix operators on BK spaces [1–4]. For a broader functional-analytic treatment of measures of non-compactness and their applications, see Rakočević [14].

When the target has a Schauder basis, the measure of non-compactness can often be read from tail projections. Let $P_m:c_0\to c_0$ be the coordinate projection $P_mz=(z_0,\dots,z_m,0,0,\dots)$. For bounded $Q\subset c_0$,

$$\chi_{c_0}(Q)=\lim_{m\to\infty} \sup_{z\in Q} \|(I-P_m)z\|_\infty. \quad (2.3)$$

The same identity holds in ℓ_r, $1\le r<\infty$, with the ℓ_r norm. For c, write $L(z)=\lim_n z_n$ and let $P_m^c z=L(z)e+\sum_{n=0}^{m}(z_n-L(z))e^{(n)}$. Then

$$(1/2)L_Q \le \chi_c(Q) \le L_Q,\quad L_Q=\lim_{m\to\infty}\sup_{z\in Q}\|(I-P_m^c)z\|_\infty. \qquad (2.4)$$

The factor 1/2 in (2.4) reflects the geometry of c and is standard in matrix-operator estimates based on the Hausdorff measure [1,2]. For compactness, however, the factor is immaterial: $\chi_c(Q)=0$ exactly when $L_Q=0$.

2.1. Operator measure of non-compactness and basic inequalities

For a bounded operator $T:E\to F$ it is convenient to write $\|T\|_\chi=\chi_F(T(B_E))$, where B_E is the closed unit ball. This quantity is homogeneous and vanishes exactly for compact operators. It is not, in general, an operator norm, but it behaves well under composition: if $R:F\to G$ and $S:D\to E$ are bounded, then

$$\|RTS\|_\chi \le \|R\|\,\|T\|_\chi\,\|S\|. \qquad (2.5)$$

Indeed, Lipschitz maps scale Hausdorff measures by at most their Lipschitz constant, and $S(B_D)\subset\|S\|B_E$. This elementary inequality is the quantitative reason the matrix-domain isometry is so effective. When the intertwining maps have norm one in both directions, the two inequalities obtained by transferring forward and backward become equalities.

Another useful property is invariance under closure and finite unions: $\chi(Q)=\chi(\text{cl } Q)$, and adjoining a relatively compact set does not change the measure. Consequently, if K is compact, then $\|T+K\|_\chi=\|T\|_\chi$ whenever T and K have the same domain and range. To see this, note that $(T+K)(B_E)$ is contained in $T(B_E)+K(B_E)$, with the second set relatively compact, and use the standard stability of χ under Minkowski addition by relatively compact sets; reverse the argument with $T=(T+K)-K$.

This compact-perturbation invariance makes $\|\cdot\|_\chi$ a natural quantity for essential operator theory. In the matrix-domain setting it can be evaluated on the model operator and therefore separates the compactness issue from the coordinate complexity introduced by A.

3. Transfer through a matrix domain

Let X be a Banach sequence space, A a triangle and X_A its matrix domain with the natural norm. Put $U_A x=Ax$ and $V=A^{-1}$. Then U_A is an isometric isomorphism from X_A onto X. Isometries preserve covering radii and therefore preserve the Hausdorff measure of non-compactness of bounded sets. This use of an invertible triangle as a coordinate isomorphism is consistent with the matrix-domain framework in Malkowsky and Rakočević [11] and the compact-operator treatment of matrix domains by Djolović and Malkowsky [13].

Theorem 3.1. For every bounded $Q\subset X_A$, $\chi_{X_A}(Q)=\chi_X(AQ)$.

Proof. For $\varepsilon>0$, a finite set $\{x^1,\dots,x^N\}$ is an ε-net for Q in X_A exactly when $\{Ax^1,\dots,Ax^N\}$ is an ε-net for AQ in X, because $\|x-x^j\|_{X_A}=\|A(x-x^j)\|_X$. Taking the infimum over ε gives the identity.

Theorem 3.2 (Domain transfer). Let $T\in B(X_A,Y)$ and define $\hat{T}=T\,U_A^{-1}:X\to Y$. Then $\|T\|_\chi=\|\hat{T}\|_\chi$. Consequently, T is compact if and only if $\hat{T}$ is compact.

Proof. The isometry U_A maps B_A onto B_X. Hence $T(B_A)=\hat{T}(B_X)$, so their Hausdorff measures in Y are identical.

If T is represented by an infinite matrix $B=(b_{nk})$, the associated operator $\hat{T}$ is represented, under the usual convergence assumptions, by the associated matrix $\hat{B}=BV$. Its entries are

$$\hat{b}_{nj}=\sum_{k=j}^{\infty} b_{nk}\,v_{kj}. \tag{3.1}$$

Thus all compactness criteria below depend on B only through the inverse-domain transform (3.1). For endomorphisms $T:X_A\to X_A$, the target must also be transferred; then the conjugate operator $C=U_ATU_A^{-1}=ATV$ acts on X.

Corollary 3.3. If $T\in B(X_A)$, then $\|T\|_\chi=\|U_ATU_A^{-1}\|_\chi$. In particular, T is compact on X_A if and only if $U_ATU_A^{-1}$ is compact on X.

3.1. Quantitative invariance under the domain isometry

The basic transfer theorem can be strengthened to an equality for operator measures of non-compactness. Let $U_A:X_A\to X$ be the isometry $U_Ax=Ax$. If $T:X_A\to Y$ is bounded, put $\hat{T}=T\,U_A^{-1}:X\to Y$. Since U_A maps the unit ball of X_A exactly onto the unit ball of X,

$$\|T\|_\chi=\chi_Y(T(B_{X_A}))=\chi_Y(\hat{T}(B_X))=\|\hat{T}\|_\chi. \tag{3.2}$$

For an endomorphism $T:X_A\to X_A$, define $C=U_ATU_A^{-1}$. Applying U_A also to the image set gives $\|T\|_\chi=\|C\|_\chi$. Thus both compactness and its quantitative defect are unchanged by the matrix-domain representation.

Corollary 3.4. If $B:X_A\to Y$ is represented by a matrix and the associated matrix $\hat{B}=BA^{-1}$ defines a bounded map $X\to Y$, then B is compact if and only if $\hat{B}$ is compact, and their operator measures of non-compactness coincide.

The formula has a practical implication. Any estimate for χ established on a familiar model space may be transplanted without worsening its constant. This is stronger than merely knowing that compactness is preserved by Banach-space isomorphism, where the product of the norms of the intertwining maps would generally enter the estimate.

4. Operators from $\ell_p(A)$ into c_0 and c

Let $1\le p<\infty$ and let q be the conjugate exponent, with $q=\infty$ when $p=1$. Suppose B defines a bounded matrix operator from $\ell_p(A)$ into c_0. Let $\hat{B}=BA^{-1}$ and write $\hat{b}_n=(\hat{b}_{nk})_k$ for its nth row. Since $(\ell_p)^*=\ell_q$, the norm of the nth coordinate functional is $\|\hat{b}_n\|_q$. The tail formula (2.3) then gives an exact measure of non-compactness.

Theorem 4.1. If $B\in(\ell_p(A),c_0)$, then

$$\|L_B\|_\chi=\lim_{m\to\infty}\sup_{n>m}\|\hat{b}_n\|_q=\limsup_{n\to\infty}\|\hat{b}_n\|_q. \tag{4.1}$$

Proof. By Theorem 3.2 it is enough to consider $L_{\hat{B}}:\ell_p\to c_0$. Applying (2.3) to $Q=L_{\hat{B}}(B_{\ell_p})$ gives $\sup_{\|y\|_p\le 1}\sup_{n>m}|\sum_k\hat{b}_{nk}y_k|$. Interchanging the two suprema and using the dual norm on ℓ_p yields $\sup_{n>m}\|\hat{b}_n\|_q$. Letting $m\to\infty$ proves (4.1).

Corollary 4.2. A bounded matrix operator $B:\ell_p(A)\to c_0$ is compact if and only if $\|\hat{b}_n\|_q\to 0$.

The boundedness assumption includes the usual c_0 mapping condition: each coordinate column of $\hat{B}$ must tend to zero in n, and $\sup_n\|\hat{b}_n\|_q<\infty$. Compactness strengthens boundedness by requiring the entire row functionals to vanish in the dual norm, not merely coordinatewise.

4.1. Target c

Now assume $B\in(\ell_p(A),c)$. For each k, the column limit $\alpha_k=\lim_{n\to\infty}\hat{b}_{nk}$ exists. The limit functional $y\mapsto\lim_n(\hat{B}y)_n$ is represented by $\alpha=(\alpha_k)\in\ell_q$. Set $r_n=\hat{b}_n-\alpha$. The projection estimate (2.4) gives the following.

Theorem 4.3. If $B\in(\ell_p(A),c)$, then

$(1/2)\ \limsup_{n\to\infty}\|\hat{b}_n-\alpha\|_q \le \|L_B\|_\chi \le \limsup_{n\to\infty}\|\hat{b}_n-\alpha\|_q$. (4.2)

Consequently, $L_B:\ell_p(A)\to c$ is compact if and only if $\|\hat{b}_n-\alpha\|_q\to0$. The criterion has a clear interpretation: the row functionals of the associated matrix must converge in the dual norm to the functional that gives the limit of the output sequence. Mere coordinatewise convergence of the rows is not enough for compactness.

4.2. Essential norm and tail distance for c_0 targets

The same transformed rows also determine the distance from B to the compact operators. Let $\|T\|_e=\inf\{\|T-K\|$: K is compact$\}$ denote the essential norm. For a bounded operator $S:E\to c_0$ and the canonical coordinate projections P_m, the operators P_mS have finite-dimensional range, while $Q_m=I-P_m$ converges uniformly to zero on compact subsets of c_0.

Theorem 4.4. Let $1\le p<\infty$, let q be conjugate to p for p>1 ($q=\infty$ for p=1), and let $B:\ell p(A)\to c_0$ be bounded. If $\hat{b}_n$ is the nth row of BA^{-1}, then

$\|B\|_e = \lim_m\to\infty\ \|Q_mBA^{-1}\| = \limsup_n\to\infty\ \|\hat{b}_n\|q = \|B\|\chi$. (4.3)

Proof. Since P_mBA^{-1} has finite rank, $\|B\|_e=\|BA^{-1}\|_e\le\liminf_m\|Q_mBA^{-1}\|$. Conversely, for any compact $K:\ell p\to c_0$, uniform convergence $Q_mK\to0$ on $K(B\ell p)$ gives $\limsup_m\|Q_mBA^{-1}\|\le\|BA^{-1}-K\|$. Taking the infimum over compact K yields equality with the tail norm. Hölder duality identifies $\|Q_mBA^{-1}\|$ with $\sup_{n>m}\|\hat{b}_n\|q$, and Theorem 4.1 identifies the same limit with the Hausdorff operator measure.

Thus, for c_0 targets, compactness, vanishing Hausdorff measure of non-compactness and zero essential norm are encoded by exactly the same transformed-row tail. The limiting row size is therefore the precise norm-distance from B to the compact operators, not merely a yes-or-no compactness test.

4.3. Convergent targets and the limiting functional

For $B:\ell p(A)\to c$, boundedness implies that for each fixed k the transformed coefficients $\hat{b}_{nk}$ converge to some a_k and that $a=(a_k)$ defines a continuous functional on ℓp. Write $a(y)=\sum a_ky_k$. The noncompact part is measured by the deviation of the rows from a. Standard projection estimates on c give

$(1/2)\ \limsup_{n\to\infty}\|\hat{b}_n-a\|_q \le \|B\|_\chi \le \limsup_{n\to\infty}\|\hat{b}_n-a\|_q$. (4.4)

Consequently B is compact if and only if $\|\hat{b}n-a\|q\to0$. The factor 1/2 in the lower estimate reflects the fact that the canonical projections on c must separate the one-dimensional limit direction from the null tail. In the special case a=0 the range is contained in c_0 and the exact c_0 formula (4.1) is recovered.

5. Operators into ℓ_r and tail matrix norms

Let $1\le r<\infty$ and suppose $T\in B(X_A,\ell_r)$. Write $\hat{T}=T\ U_A^{-1}:X\to\ell_r$. Let $Q_m:\ell_r\to\ell_r$ denote the tail projection $Q_mz=(0,\dots,0,z_{m+1},z_{m+2},\dots)$. The exact tail formula for ℓ_r gives

Theorem 5.1. For $T\in B(X_A,\ell_r)$,

$$\|T\|_\chi = \lim_{m\to\infty} \|Q_m\ \hat{T}\|_{B(X,\ell r)}. \quad (5.1)$$

Proof. Apply Theorem 3.2 and the ℓ_r version of (2.3) to $\hat{T}(B_X)$. The quantity $\sup_{y\in B_X}\|Q_m\hat{T}y\|r$ is exactly the operator norm $\|Q_m\hat{T}\|$.

Corollary 5.2. $T:X_A\to\ell_r$ is compact if and only if $\|Q_m\hat{T}\|\to0$.

For a matrix B, $Q_m\hat{B}$ is obtained by deleting the first m+1 rows and retaining the infinite tail. Thus compactness is equivalent to the operator norm of the row tail tending to zero. The condition is abstract but becomes completely explicit for several domain spaces.

5.1. Explicit criterion for $\ell_1(A)$

Take $X=\ell_1$. For a matrix $M=(m_{nk})$ acting from ℓ_1 into ℓ_r, one has $\|M\|_{\ell_1\to\ell_r}=\sup_k(\sum_n|m_{nk}|^r)^{1/r}$, because the unit ball of ℓ_1 is the closed absolutely convex hull of the coordinate vectors. Therefore Theorem 5.1 yields an entrywise column-tail criterion.

Theorem 5.3. Let $B:\ell_1(A)\to\ell_r$ be bounded and $\hat{B}=BA^{-1}$. Then

$$\|L_B\|_\chi = \lim_{m\to\infty} \sup_k \left[\sum_{n>m}|\hat{b}_{nk}|^r\right]^{1/r}. \quad (5.2)$$

Hence B is compact if and only if the right-hand side of (5.2) is zero. The condition is uniform in the column index k; pointwise decay of each column tail is not sufficient without this uniformity.

5.2. A Hilbert-space sufficient condition

When $X=\ell_2$ and the target is ℓ_2, a simple sufficient condition follows from the Hilbert-Schmidt class. If the associated matrix satisfies $\sum_{n,k}|\hat{b}_{nk}|^2<\infty$, then $L_{\hat{B}}$ is Hilbert-Schmidt and therefore compact on ℓ_2. By Theorem 3.2, $B:\ell_2(A)\to\ell_2$ is compact. This criterion is not necessary for compactness, but it is easy to verify and often useful for kernels with polynomial or exponential decay.

5.3. Tail norms for maps into ℓr

Let $1\le r<\infty$ and Q_m be the tail projection on ℓr. Since ℓr has the AK property and coordinate projections converge strongly to the identity, a bounded set $Q\subset\ell r$ is relatively compact exactly when its tails vanish uniformly. For $T:X\to\ell r$ this yields

$$\|T\|_\chi = \lim_{m\to\infty} \|Q_m T\|. \quad (5.3)$$

Combining with the domain isometry gives $\|B\|_\chi=\lim_m\|Q_mBA^{-1}\|$ for $B:X_A\to\ell r$. Hence B is compact if and only if the tail matrices Q_mBA^{-1} converge to zero in operator norm. This criterion is abstractly simple but can be specialized by known matrix norms.

For $B:\ell 1(A)\to\ell r$, the operator norm of a matrix $C=(c_{nk})$ from $\ell 1$ to ℓr is $\sup_k(\sum_n|c_{nk}|^r)^{1/r}$. Therefore

$$\|B\|_\chi = \lim_{m\to\infty} \sup_k \left[\sum_{n>m}|\hat{b}_{nk}|^r\right]^{1/r}. \quad (5.4)$$

This is an exact uniform column-tail formula. It shows that compactness from $\ell 1(A)$ into ℓr requires the transformed columns to have ℓr tails tending to zero uniformly in the column index. Pointwise decay of each individual column is insufficient; the uniformity is essential.

5.4. Sufficient Schur-type estimates for $1<p<\infty$

For general p and r, exact matrix norms may be difficult to compute. Nevertheless, elementary majorization can provide useful sufficient compactness conditions. For example, suppose $1<p<\infty$ and the transformed tail matrix $C^{(m)}=Q_mBA^{-1}$ satisfies row and column estimates that make a Schur test applicable. If numbers R_m and C_m tend to zero in such a way that $\sup_{n>m}\sum_k|c_{nk}|\le R_m$ and $\sup_k\sum_{n>m}|c_{nk}|\le C_m$, then on $\ell 2$ one obtains $\|C^{(m)}\|\le(R_mC_m)^{1/2}\to 0$.

More generally, interpolation between endpoint estimates can yield decay of $\|Q_mBA^{-1}\|_{\ell p\to\ell p}$ when the same tail matrix is controlled on $\ell 1$ and $\ell\infty$. These estimates are only sufficient, not necessary, but they are often easier to verify for kernels defined by explicit formulas. The central methodological point remains the same: all such estimates are applied to BA^{-1}, not to B alone.

6. Compact endomorphisms of X_A

Suppose now that T maps X_A into itself. The correct model operator is the similarity $C=U_ATU_A^{-1}=AT\,A^{-1}$. Corollary 3.3 shows that compactness is invariant under this conjugation. Thus the compact ideal $K(X_A)$ is carried isometrically onto $K(X)$ by $T\mapsto U_ATU_A^{-1}$.

Theorem 6.1. The quotient Banach algebras $B(X_A)/K(X_A)$ and $B(X)/K(X)$ are isometrically isomorphic under the map induced by $T\mapsto U_ATU_A^{-1}$. Consequently, the essential norm satisfies

$$\|T\|_e = \|U_ATU_A^{-1}\|_e. \quad (6.1)$$

Proof. The similarity is an isometric algebra isomorphism and maps compact operators onto compact operators. Taking the quotient by the compact ideals preserves the norm of cosets, giving (6.1).

The essential norm and the Hausdorff

measure of non-compactness are related quantities, although they are not identical in every Banach space. Formula (6.1) nevertheless reinforces the same principle: compactness questions on a matrix domain should first be transferred to the model space. In spectral applications this also implies invariance of the Fredholm essential spectrum under the matrix-domain similarity.

6.1. Compact perturbations, essential behavior and the quotient picture

The isometric algebra isomorphism $T \mapsto U_A T U_A^{-1}$ sends $K(X_A)$ onto $K(X)$. Hence it descends to an isometric isomorphism of the Calkin algebras. In addition to the essential-norm identity already recorded, the operator measure of non-compactness satisfies

$$\|T+K\|_\chi = \|T\|_\chi \quad \text{for every } K \in K(X_A). \tag{6.2}$$

Thus $\|T\|_\chi$ depends only on the coset of T modulo compact operators. If $C=U_A T U_A^{-1}=D+K$ with D a simple diagonal or shift-like operator and K compact, then the noncompactness measure of T equals that of D. This can be easier to compute than the essential norm and can supply lower bounds for approximation by finite-rank operators.

The quotient viewpoint also clarifies why compactness criteria are stable under changes in a finite number of rows or columns of an associated matrix. Such changes frequently produce finite-rank operators after the mapping class has been verified, and finite-rank perturbations do not alter χ. Hence only the asymptotic transformed tails matter.

6.2. Approximation numbers under matrix-domain isometries

A second quantitative compactness scale is provided by approximation numbers. For a bounded operator $T:E \to F$ define $a_n(T)=\inf\{\|T-R\|: \text{rank } R<n\}$. The sequence $a_n(T)$ is nonincreasing. If $a_n(T) \to 0$, then T is compact because it is a norm limit of finite-rank operators. The converse need not hold on arbitrary Banach spaces; however, it does hold for compact operators with range in the classical spaces c_0 and ℓp $(1 \le p < \infty)$, whose canonical finite-coordinate projections yield finite-rank approximations uniformly on compact subsets.

Proposition 6.2. Let $T:X_A \to Y$ and $\hat{T}=TU_A^{-1}:X \to Y$. Then $a_n(T)=a_n(\hat{T})$ for every n. If T is an endomorphism and $C=U_A T U_A^{-1}$, then $a_n(T)=a_n(C)$.

Proof. If $R:X_A \to Y$ has rank<n, then $\hat{R}=RU_A^{-1}:X \to Y$ has the same rank and $\|T-R\|=\|\hat{T}-\hat{R}\|$ because U_A is an onto isometry. Taking infima gives $a_n(T) \ge a_n(\hat{T})$; applying the same argument to U_A^{-1} gives the reverse inequality. The endomorphism case follows by also conjugating the range.

Thus rates of compact approximation are completely unchanged by passing to a matrix domain. If the associated operator is Hilbert-Schmidt on $\ell 2$, its singular-value estimates immediately become approximation-number estimates for the domain operator. Similarly, decay rates for diagonal multipliers on the model space are inherited exactly.

7. Examples

7.1. A weighted first-difference domain

Let $A=I-\theta S$ with $|\theta|<1$, so $A^{-1}=V$ has entries $v_{nk}=\theta^{n-k}$ for $k \le n$. Consider the domain $\ell_p(A)$ and a diagonal-output matrix B given by $(Bx)_n=\gamma_n x_n$, where $\gamma=(\gamma_n)$ is bounded. The associated matrix $\hat{B}=BV$ has rows

$$\hat{b}_{nk} = \gamma_n \theta^{n-k}, \quad 0 \le k \le n; \qquad \hat{b}_{nk}=0, \ k>n. \tag{7.1}$$

For $1<p<\infty$ with conjugate q,

$$\|\hat{b}_n\|_q = |\gamma_n| \left[\sum_{j=0}^{n} |\theta|^{qj}\right]^{1/q} = |\gamma_n|[(1-|\theta|^{q(n+1)})/(1-|\theta|^q)]^{1/q}. \tag{7.2}$$

For p=1, q=∞ and $\|\hat{b}_n\|_\infty=|\gamma_n|$. If B maps $\ell_p(A)$ into c_0, Theorem 4.1 gives compactness exactly when $\gamma_n \to 0$. Indeed, because $|\theta|<1$, the geometric factor in (7.2) is bounded above and below by positive constants independent of n. This example shows how the inverse triangle changes each row but does not alter the ultimate compactness condition for a decaying diagonal multiplier.

7.2. Difference domains at the critical ratio

If $\theta=1$, then $A=I-S$ is the classical first difference and $v_{nk}=1$ for $k\leq n$. For the same diagonal-output B, the associated row consists of γ_n repeated $n+1$ times. Hence for $1<p<\infty$, $\|\hat{b}_n\|_q=|\gamma_n|(n+1)^{1/q}$. Compactness into c_0 therefore requires the stronger decay $|\gamma_n|(n+1)^{1/q}\to 0$. The contrast with $|\theta|<1$ illustrates how the growth of A^{-1} directly controls compactness on the matrix domain. The underlying first-difference sequence-space construction originates with Kızmaz [10].

For $p=1$, the dual exponent is $q=\infty$ and the condition remains $\gamma_n\to 0$. Thus the same domain can exhibit different compactness thresholds depending on the underlying ℓ_p geometry. Such distinctions are difficult to see from the original diagonal matrix B alone; they emerge immediately after multiplication by A^{-1}.

7.3. Hilbert-Schmidt kernel on $\ell_2(A)$

Let $C=(c_{nk})$ on ℓ_2 be defined by $c_{nk}=(n+k+1)^{-2}$. Since $\sum_{n,k}(n+k+1)^{-4}<\infty$, C is Hilbert-Schmidt and compact. For any triangle A defining $\ell_2(A)$, set $T=A^{-1}CA$. Then T is compact on $\ell_2(A)$, and $\|T\|_\chi=\|C\|_\chi=0$. The matrix of T can be much denser than C because both A and A^{-1} enter, but no direct compactness proof in those coordinates is required.

7.4. Finite-rank approximation

Let P_m be the model-space coordinate projection and define $T_m=A^{-1}P_mA$ on X_A. Each T_m has finite rank. If $X=c_0$ or ℓ_p, $1\leq p<\infty$, then $P_m\to I$ strongly and therefore $T_mx\to x$ for every $x\in X_A$. For a compact operator T on X_A, the standard approximation argument gives $\|(I-T_m)T\|\to 0$. In model coordinates this is simply the vanishing of the tails of the compact set $U_AT(B_A)$. This recovers the projection mechanism underlying formulas (4.1) and (5.1).

7.5. Product domains: a weighted mean followed by a difference triangle

Many sequence spaces in the literature are generated not by a single elementary triangle but by a product $A=G\Delta^m$, where G is a generalized weighted mean. The inverse then factors as $A^{-1}=\Delta^{-m}G^{-1}$. For a matrix $B:X_A\to Y$, the associated matrix is

$$\hat{B} = BA^{-1}=B\Delta^{-m}G^{-1}. \tag{7.3}$$

The factorization permits compactness estimates to be organized in two stages. First the rows of B are transformed by repeated discrete summation Δ^{-m}, which introduces polynomial binomial weights. Then G^{-1} applies the correction coming from the weighted mean. If both inverse factors have explicit triangular coefficients, the resulting associated rows can be written as nested but finite lower-triangular sums. Closely related generalized-mean difference constructions and compactness criteria appear in Maji, Manna and Srivastava [8] and Mursaleen and Noman [12].

For c_0 targets, Corollary 4.2 states that compactness is equivalent to the q-norms of these doubly transformed rows tending to zero. This recovers, at an abstract level, the structure of compactness tests obtained for generalized-mean difference spaces: the complicated coefficient conditions are exactly the ordinary c_0 row criterion applied after the inverse product has absorbed the domain constraint.

7.6. Diagonal multipliers on weighted difference domains

Let $A=I-\theta S$ with $|\theta|\leq 1$ and let B be the coordinate multiplier $(Bx)_n=\gamma_nx_n$ acting from $\ell p(A)$ to c_0. As in (7.1), $\hat{b}_{nk}=\gamma_n\theta^{n-k}$ for $k\leq n$. Therefore Theorem 4.1 gives

$$\|B\|_\chi = \limsup_{n\to\infty}|\gamma_n| \left[\sum_{j=0}^{n}|\theta|^{qj}\right]^{1/q}, \tag{7.4}$$

with the obvious supremum interpretation when $p=1$. If $|\theta|<1$, the bracket converges to a finite positive constant and compactness is equivalent to $\gamma_n\to 0$. At $\theta=1$ and $1<p<\infty$, the bracket equals $(n+1)^{1/q}$, so the exact noncompactness measure detects the strengthened threshold $|\gamma_n|(n+1)^{1/q}\to 0$.

This formula is more informative than a yes/no compactness test. For example, if $\gamma_n=(n+1)^{-a}$ and $\theta=1$, then the tail measure behaves like $\limsup(n+1)^{1/q-a}$. The operator is compact precisely when $a>1/q$; it is bounded but noncompact at the critical exponent when the transformed row norms remain bounded away from zero.

7.7. Approximation by finite-rank transformed projections

Let $X=c_0$ or ℓp, $1\leq p<\infty$, and P_m be the coordinate projection on X. Define $F_m=U_A^{-1}P_mU_A$ on X_A. Each F_m has finite rank and $\|F_m\|=1$. For a bounded endomorphism T of X_A,

$$\|(I-F_m)T\| = \|(I-P_m)U_ATU_A^{-1}\|. \qquad (7.5)$$

If T is compact, the right-hand side tends to zero because P_m converges uniformly on compact subsets of X. Conversely, norm convergence of these finite-rank approximants implies compactness. Thus compactness can be recognized by a canonical domain-adapted approximation scheme, not by truncating the original coordinates of x but by truncating Ax.

The distinction matters for difference domains. Ordinary coordinate truncation of x creates artificial jumps and can have large difference norm, whereas F_m truncates the transformed sequence and reconstructs it through A^{-1}. The latter is exactly aligned with the Banach-space geometry and therefore supplies the correct finite-rank approximation mechanism.

7.8. Quantitative finite-rank error for diagonal models

Consider $C=M_d$ on ℓp or c_0 with $d_n\to0$ and $T=A^{-1}M_dA$ on X_A. Let P_m be the model coordinate projection and $R_m=A^{-1}P_mM_dA$. Then rank $R_m\leq m+1$ and

$$\|T-R_m\| = \|(I-P_m)M_d\| = \sup_{n>m}|d_n|. \qquad (7.6)$$

Consequently $a_{m+2}(T)\leq\sup_{n>m}|d_n|$. In many diagonal settings this upper bound is sharp up to the indexing convention. If d_n decays polynomially or exponentially, the same rate describes finite-rank approximability on every triangular matrix domain built over the model space.

This example complements the Hausdorff measure. The measure decides whether the tail ultimately vanishes, whereas approximation numbers describe how rapidly compactness is achieved by finite-rank operators. Because both are evaluated after the same isometric transform, a single associated matrix can support qualitative and quantitative compactness analysis simultaneously.

7.9. Compactness of weighted shift type operators

Let $X=\ell p$, $1\leq p<\infty$, and let W be a weighted right shift on X, $(Wy)_0=0$ and $(Wy)_n=w_{n-1}y_{n-1}$. Define $T=A^{-1}WA$ on X_A. Since compactness and the Hausdorff measure are invariant under the domain isometry, T is compact if and only if W is compact. For a unilateral weighted shift on ℓp, compactness is equivalent to $w_n\to0$.

Indeed, $W e^{(n)}=w_n e^{(n+1)}$, so compactness forces $\|W e^{(n)}\|=|w_n|\to0$ because the coordinate vectors form a weakly null bounded sequence when $1<p<\infty$, and a direct disjoint-support argument gives the same necessity for p=1. Conversely, if $w_n\to0$, the finite-rank truncations that retain only the first m weights converge to W in operator norm.

Consequently every transported weighted shift $T=A^{-1}WA$ is compact exactly when the model weights vanish. The raw matrix of T may combine the shift with both A and A^{-1}, but its compactness threshold remains the simple scalar condition on w. Moreover, the approximation error after retaining the first m weights is $\sup_{n>m}|w_n|$, and this rate transfers exactly to X_A.

7.10. Relative compactness of bounded families

The same machinery applies to families of vectors rather than operators. Let $Q\subset X_A$ be bounded. Since U_A is an isometry, $\chi_{X_A}(Q)=\chi_X(U_AQ)$. Thus Q is relatively compact if and only if its transformed family AQ is relatively compact in X. For $X=c_0$ or ℓp, this means that compactness of Q can be tested by uniform decay of the transformed tails.

$$\lim_{m\to\infty} \sup_{x\in Q} \|Q_m(Ax)\|_X = 0 \qquad (7.7)$$

together with boundedness, characterizes relative compactness for these classical sequence spaces. This criterion can be easier to use than any direct condition on the original coordinates x, especially in difference domains where x itself may grow while Ax has a small tail.

For example, a bounded family in $\ell p(\Delta^m)$ may contain sequences with large polynomial-looking prefixes or slow pointwise decay. Such behavior does not prevent relative compactness if the mth differences have uniformly vanishing

ℓp tails. The domain norm deliberately regards the transformed sequence as the true coordinate variable, and the Hausdorff measure respects that geometry exactly.

7.11. Compactness under composition of domain operators

Let $T:X_A \to Y_B$ and $S:Y_B \to Z_C$ be bounded operators between three triangular matrix domains. Put $\tilde{T}=BTA^{-1}:X\to Y$ and $\tilde{S}=CSB^{-1}:Y\to Z$. Then the model representative of the composition is

$$C(ST)A^{-1} = (CSB^{-1})(BTA^{-1}) = \tilde{S}\tilde{T}. \quad (7.8)$$

Hence if either T or S is compact, ST is compact, and quantitative estimates can be performed entirely on the product $\tilde{S}\tilde{T}$. This extends the single-domain transfer principle to chains of matrix-domain spaces and shows that the category of such domains with bounded operators is isometrically equivalent, at the operator level, to the corresponding category of model spaces.

The formulation is useful when one operator is naturally expressed in one transform and another operator in a different transform. Rather than multiplying raw infinite matrices first, one can convert each map to its model representation, compose there, and then reconstruct if an explicit domain matrix is actually needed. Compactness, finite rank and approximation estimates survive this procedure unchanged.

7.12. A compactness checklist for concrete matrix domains

For a concrete matrix B acting from X_A, the compactness analysis can be reduced to four steps. First determine the inverse triangle A^{-1}. Second form the associated operator BA^{-1} when the range is an ordinary sequence space, or ABA^{-1} for an endomorphism. Third verify boundedness on the model spaces using standard matrix criteria. Fourth apply the tail formula appropriate to the target: row tails for c_0, deviations from the limiting row for c, or operator-norm tails for ℓr.

This sequence of steps is preferable to testing compactness directly on arbitrary bounded sequences in X_A. The unit ball of X_A may be difficult to visualize in original coordinates, while A maps it exactly onto the familiar unit ball of X. The Hausdorff measure then converts relative compactness into a tail problem that is usually explicit.

The same checklist also distinguishes bounded but noncompact operators. If transformed tails remain uniformly bounded yet fail to vanish, the operator is bounded and its nonzero limiting tail size gives the quantitative defect of compactness. Thus the method does not stop at a binary compact/noncompact conclusion; it records how far the operator remains from compact behavior in the asymptotic coordinates.

Because compact perturbations and finite changes do not affect the asymptotic measure, attention can be focused on the stable tail pattern of the associated matrix. This is particularly useful for difference and weighted-mean domains, where the first several rows often contain complicated boundary terms that have no bearing on compactness.

8. Concluding remarks

The Hausdorff measure of non-compactness fits matrix domains particularly well because the natural domain norm makes the defining triangle an isometry. As a result, compactness is not an intrinsically new problem on X_A: it is the compactness of an associated operator on X. What changes is the matrix representation. For maps out of the domain, the rows must be transformed by A^{-1}; for endomorphisms, the full similarity $A\,T\,A^{-1}$ is required.

This transfer principle yields exact and usable criteria. For $B:\ell_p(A)\to c_0$, compactness is equivalent to the dual norms of the associated rows tending to zero. For maps into c, the rows must converge in the dual norm to the limiting functional. For targets ℓ_r, compactness is equivalent to the vanishing of tail operator norms, and on $\ell_1(A)$ this becomes a uniform column-tail condition. The weighted-difference examples show explicitly how growth in A^{-1} can strengthen the decay required of the original coefficients.

The framework is suitable for further

extensions to fractional difference triangles, generalized weighted means, special-number matrices and other domains currently used in sequence-space research. Once the inverse triangle is known, the central compactness calculations can be organized around the associated matrix rather than repeated from first principles for each newly named space.

9. Extensions and open directions

The transfer formulas suggest several extensions. One is to domains generated by products of triangles. If $A=A_1A_2$, then X_A can be reached through two successive isometries, and an operator $B:X_A\to Y$ has associated model operator $BA^{-1}=BA_2^{-1}A_1^{-1}$. This factorization can be exploited when one factor is a generalized mean and the other is a difference operator, a common construction in contemporary sequence-space research. Each factor can be handled separately, which often simplifies coefficient estimates.

Another direction is the comparison of the Hausdorff measure with the essential norm. On c_0 and ℓ_p, tail projections frequently give both quantities in closely related forms. For concrete matrix classes it is therefore natural to ask when $\|T\|_\chi=\|T\|_e$, or when sharp constants relate them. Because both quantities are invariant under the isometric domain transform, any equality proved on the model space immediately yields the corresponding equality on X_A.

The framework also extends to block sequence spaces and vector-valued entries. If X is replaced by $\ell_p(E)$ for a Banach space E and A acts on the scalar sequence index, the same domain isometry remains available. Compactness then depends on both the tail behavior in the sequence index and compactness properties of the operator-valued matrix entries. This provides a route from scalar summability matrices to discrete systems of operators and evolution equations.

From a computational viewpoint, Theorem 5.1 offers a direct diagnostic: approximate the norm of the tail matrix Q_mBA^{-1} for increasing m. A persistent positive tail norm signals non-compactness, while decay toward zero supports compactness. For Hilbert-space models, singular-value or Hilbert-Schmidt estimates can be applied after the same transformation. The important point is that numerical or analytic effort should be spent on the associated matrix, where the domain constraint has already been absorbed.

Future work may combine these compactness criteria with similarity-based spectral transfer on matrix domains. If $ABA^{-1}=D+K$ with D diagonal or shift-like and K compact, then the Fredholm essential spectrum agrees with that of D. On a connected component of its complement where the Fredholm index is zero and which contains a resolvent point, analytic Fredholm theory implies that any additional spectral points are isolated eigenvalues of finite algebraic multiplicity. Matrix domains therefore provide a natural setting in which summability, compactness and spectral theory can be integrated through a single change of coordinates.

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