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Matrix Domains of Generalized Difference Operators on Model Spaces

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Abstract **Original Research Article**

This paper develops a unified matrix-domain framework for higher-order generalized difference operators on the classical model sequence spaces $c_0, c,\ \ell_p$ andℓ_∞. For nonzero α and arbitrary β, the mth-order generalized difference triangle $D_\alpha,\beta^{(m)} = (\alpha I + \beta S)^m$ is introduced, where S is the unilateral lower shift. An explicit inverse triangle is obtained from the negative-binomial expansion. Using this inverse, the matrix domains $X(D_\alpha,\beta^{(m)})$ are shown to be BK spaces and to be isometrically isomorphic to their underlying model spaces. For spaces with the AK property, a Schauder basis is constructed directly from the columns of the inverse triangle. Transfer descriptions of the $\alpha-,\beta-$ $and\ \gamma$-duals and a reduction theorem for matrix transformations from $X(D_\alpha,\beta^{(m)})$ into an arbitrary sequence space are established. The classical m$^{\text{th}}$-order difference spaces arise when $\alpha = 1$ and $\beta = -1$, while weighted differences and two-band difference operators occur as special cases. The results isolate the operator-theoretic mechanism behind many individually defined difference sequence spaces and provide a reusable starting point for spectral and compactness questions.

Keywords: matrix domain; generalized difference operator; BK space; Schauder basis; sequence space; matrix transformation.

1. Introduction

Difference sequence spaces form one of the most persistent interfaces between summability theory and functional analysis. The basic idea is simple: rather than requiring a sequence $x = (x_k)$ to belong directly to a classical sequence space X, one requires a difference transform of x to belong to X. In the first-order setting this leads to domains of the difference matrix Δ, and the approach can be iterated, weighted, or combined with other triangular matrices. Kızmaz initiated the systematic study of such spaces for the classical difference operator, while later work treated generalized two-band and higher-order operators, non-absolute type spaces, generalized means and related matrix domains [1-6].

The matrix-domain viewpoint provides a particularly efficient language for these constructions. If $A = (a_{nk})$ is an infinite matrix and X is a sequence space, its matrix domain in X is $X_A = \{x \in \omega: Ax \in X\}$. When A is a triangle, the map $x \mapsto Ax$ is algebraically invertible on the space ω of all scalar sequences. Under the natural norm $\|x\|_A = \|Ax\|_X$, many geometric and topological questions on X_A reduce to corresponding questions on X. This principle is implicit in a large part of the modern literature on matrix domains, including generalized difference spaces and domains generated by weighted means [3-6].

The objective here is not to introduce another isolated named sequence space, but to formulate a compact family of generalized difference triangles for which the inverse is explicit and the associated domain theory can be developed in a single argument. Let S denote the unilateral lower shift, $(Sx)_0 = 0$ and $(Sx)_n = x_{n-1}\, for\, n \geq 1$. For an integer $m \geq 1$ and scalars $\alpha \neq 0$ and β, define $D_\alpha, \beta^{(m)} = (\alpha I + \beta S)^m$. This family includes the classical mth difference $\Delta^m = (I - S)^m$, scaled differences, and repeated weighted first differences. Because $D_\alpha, \beta^{(m)}$ is a finite-band triangle with nonzero diagonal α^m, it has a unique inverse triangle. The inverse coefficients can be read from the negative-binomial series, which makes basis vectors and associated matrix transformations explicit.

Three themes are developed. First, the basic BK-space structure and isometric isomorphism of the domains are established for c_0, c, ℓ_p and ℓ_∞. Second, the columns of the inverse matrix are used to construct Schauder bases in the spaces whose underlying model has the AK property. Third, linear functionals and infinite matrix transformations are transferred from the new domain to the original model space through the inverse triangle. The last step is important because direct calculations on a matrix domain can obscure the essentially conjugate nature of the problem. The transfer formula given below is also the natural bridge to the spectral and compactness investigations developed in subsequent operator-theoretic applications.

Throughout, the scalar field is ℂ, although all results remain valid over ℝ with the obvious changes. The notation ω denotes all scalar sequences indexed by $\mathbb{N}_0 = \{0,1,2,\dots\}$; φ denotes the finitely supported sequences; $e^{(k)}$ is the kth coordinate vector; and $e = (1,1,\dots)$ is the constant sequence. For $1 \leq p < \infty, \ell p$ has its usual norm and q denotes the conjugate exponent when $p > 1$. The symbols c_0, c and ℓ_∞ have their usual meanings and are equipped with the supremum norm.

2. Preliminaries on matrix domains

A sequence space X is called a BK space if it is a Banach space and every coordinate functional $p_k(x) = x_k$ is continuous. The spaces $c_0, c,\ \ell_p$ and ℓ_∞ are standard BK spaces. A BK space X containing φ has the AK property if each $x \in X$ is the norm limit of its canonical sections $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$. Both c_0 and $\ell_p, 1 \leq p < \infty$, have AK, whereas ℓ∞ does not. The space c has a Schauder basis after separating the limit component, but its canonical coordinate vectors alone do not give AK in the above sense.

An infinite matrix $A = (a_{nk})$ is called a triangle when $a_{nn} \neq 0$ for every n and $a_{nk} = 0$ whenever $k > n$. Every triangle has a unique inverse triangle $V = A^{-1}$. If X is a BK space, the matrix domain X_A is again a BK space when it is equipped with the graph norm induced by A. In the triangular case the map $T_A: X_A \to X,\ T_A x = Ax$, is onto because V maps every $y \in \omega$ to a unique $x = Vy$. Thus, for the natural norm, T_A is an isometric isomorphism.

$$X_A = \{x \in \omega : Ax \in X\}, \qquad \| x \|_A = \| Ax \|_X. \tag{2.1}$$

For subsets $X, Y \subset \omega$, the notation (X, Y) denotes the class of infinite matrices $B = (b_{nk})$ for which $Bx \in Y$ for every $x \in X$, with each coordinate series defining $(Bx)_n$ convergent. The $\beta-$dual of X is $X^\beta = \{a \in \omega : \sum_k a_k x_k$ converges for every $x \in X\}$; the $\gamma-$dual is $X^\gamma = \{a \in \omega$:the partial sums of $\sum_k a_k x_k$ are bounded for every $x \in X\}$. These duals are useful because rows of a matrix transformation on X must lie in X^β.

The literature contains many examples in which a new sequence space is defined as the domain of a triangle and then studied by a direct coordinate computation. The present approach emphasizes a recurring structural fact: if the inverse triangle is available, basis questions, duals, operator norms and many mapping properties can be expressed in terms of a transformed matrix acting on the simpler model space. This reduction is especially transparent for generalized difference matrices because their inverse coefficients have a closed form. Classical matrix-transformation criteria and general triangle-domain theory provide the standard background for this reduction [9,10].

3. The generalized difference triangle

Fix an integer m ≥1, a scalar $\alpha \neq 0$ and $\beta \in$ ℂ. Let S be the lower shift on ω. Since I and S

commute, the binomial theorem gives $D_\alpha,\beta^{(m)} = (\alpha I + \beta S)^m$. In coordinates, $D_\alpha,\beta^{(m)}$ is an $(m+1)$ −band lower triangular matrix. Its entries are

$$d_{nk} = \binom{m}{n-k} \alpha^{m-(n-k)} \beta^{n-k}, \quad 0 \le n-k \le m; \qquad d_{nk} = 0 \text{ otherwise.} \tag{3.1}$$

Here $C(m,j) = m!/[j!\,(m-j)!]$. In particular, $d_{nn} = \alpha^m \neq 0$, so $D_\alpha,\beta^{(m)}$ is a triangle. The action on a sequence x can be written as a finite generalized difference

$$(D_{\alpha,\beta}^{(m)} x)_n = \sum_{j=0}^{\min(m,n)} \binom{m}{j} \alpha^{m-j} \beta^j x_{n-j}. \tag{3.2}$$

Theorem 3.1. The inverse $V = (D_\alpha,\beta^{(m)})^{-1}$ is the triangle with entries

$$v_{nk} = \alpha^{-m} \left(-\frac{\beta}{\alpha}\right)^{n-k} \binom{m+n-k-1}{n-k}, \quad 0 \le k \le n. \tag{3.3}$$

Proof. Factor $D_\alpha,\beta^{(m)} = \alpha^m (I + (\frac{\beta}{\alpha})S)^m$. In the algebra of formal lower triangular matrices, the negative-binomial identity

$(1+z)^{-m} \sum_{j=0}^{\infty} (-1)^j\, C(m+j-1) S^j$ yields

$(I + (\frac{\beta}{\alpha})S)^{-m} = \sum_{j\ge 0} \left(-\frac{\beta}{\alpha}\right)^j\, C(m+j-1) S^j$. Reading the coefficient of S^{n-k} gives (3.3). Multiplication of the two formal series equals I, and every entry of the product involves only finitely many terms, so the identity is valid entrywise on ω.

The explicit inverse already explains several characteristic features of difference domains. When $\beta = -\alpha$, the inverse coefficients grow polynomially like $C(m+n-k-1, n-k)$; hence elements of $X(D_\alpha, -\alpha^{(m)})$ can have polynomial growth even when their transformed sequence lies in a small model space. When $|\beta/\alpha| < 1$, the geometric factor in (3.3) suppresses the polynomial term and the inverse behaves much more regularly. Thus the ratio $|\beta/\alpha|$ controls how far the domain can extend beyond X.

For $m = 1$ the transformation reduces to $(D_{\alpha,\beta}^{(1)} x)_n = \alpha x_n + \beta x_{n-1}$, with inverse $v_{nk} = \alpha^{-1}(-\beta/\alpha)^{n-k}$. For $\alpha = 1$ and $\beta = -1$, $D_{1,-1^{(1)}}$ is the ordinary backward difference Δ with $(\Delta x)_0 = x_0$ and $(\Delta x)_n = x_n - x_{n-1}$. For general m, $D_{1,-1^{(m)}} = \Delta^m$. The formulation therefore contains both the classical difference sequence spaces and the generalized two-band domains studied in the literature [1–5].

4. Matrix domains on the model spaces

For $X \in \{c_0, c, \ell_p, \ell_\infty\}$, define the generalized difference domain

$$X\left(D_{\alpha,\beta}^{(m)}\right) = \left\{x \in \omega : D_{\alpha,\beta}^{(m)} x \in X\right\}. \tag{4.1}$$

We abbreviate $D = D_\alpha,\beta^{(m)}$ and $V = D^{-1}$. The next result gives the basic structure without any case-by-case calculation.

Theorem 4.1. Let X be any BK space. Then $X(D)$ is a BK space under $\|x\|_{X(D)} = \|Dx\|_X$, and $T_D : X(D) \to X$ defined by $T_D x = Dx$ is an isometric isomorphism. Its inverse is $T_D{}^{-1} y = Vy$.

Proof. Linearity and the norm identity are immediate. Since D is a triangle, it is bijective on ω with inverse V; therefore T_D is one-to-one and onto X. Completeness of $X(D)$ follows from completeness of X: if (x^j) is Cauchy in $X(D)$, then (Dx^j) converges to some $y \in X$, and x^j converges in the domain norm to Vy. Continuity of coordinates follows because $x_n \sum_{k=0}^{n} v_{nk}\,(Dx)_k$ is a finite linear combination of continuous coordinate functionals on X.

Theorem 4.1 has several immediate consequences. Reflexivity, separability and the presence of a Schauder basis are preserved whenever they are invariant under Banach-space isomorphism. Thus $\ell_p(D)$ is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$; $c_0(D)$ is separable and nonreflexive; and $\ell_\infty(D)$ remains nonseparable. These facts concern the Banach-space geometry, not coordinatewise inclusion, and therefore do not depend on the possibly large entries of V.

Corollary 4.2. For $1 \le p < \infty$, the spaces $\ell_p(D)$ and ℓ_p are isometrically isomorphic. Likewise, $c_0(D)$, $c(D)$ and $\ell_\infty(D)$ are isometrically isomorphic to c_0, c and ℓ_∞, respectively. In particular, none of these constructions creates a new Banach-space isomorphism class, although it generally creates a new set of sequences and a new coordinate geometry.

This distinction is important in interpreting matrix-domain constructions. A domain can be much larger than its underlying model space as a subset of ω and can possess a non-absolute norm, yet remain isometric to the original space after applying the defining triangle. Many useful operator formulas exploit precisely this change of coordinates.

4.1. Non-absolute behavior and strict extensions

For the classical difference choice $\alpha = 1, \beta = -1$, the domains are generally of non-absolute type. For example, with $m = 1\ in\ c_0(\varDelta)$, consider $x = (1, -1, 0, 0, \dots)$. Then $\|x\|_{c_0(\varDelta)} = \|(1, -2, 1, 0, \dots)\|_\infty = 2$, while $|x| = (1,1,0,0,\dots)$ satisfies $\|x\|_{c_0(\varDelta)} = \|(1, 0, -1, 0, \dots)\|_\infty = 1$. Hence the domain norm is not invariant under coordinatewise absolute value.

Proposition 4.3. For $m \geq 1$ and the classical difference $D = \varDelta^m$, one has $\ell_p \subsetneq \ell_p(\varDelta^m)$ for $1 \leq p < \infty\ and\ c_0 \subsetneq c_0(\varDelta^m)$.

Proof. The inclusion follows because $\varDelta^m$ is a finite linear combination of shifts, each bounded on ℓ_p and c_0. It is strict because the constant sequence e does not lie in ℓ_p or c_0, whereas $\varDelta^m e$ has finite support under the convention $(\varDelta x)_0 = x_0$ and therefore belongs to $\varphi \subset \ell_p \cap c_0$.

The same argument can be adapted to polynomial sequences of degree below m, whose m$^{\text{th}}$ differences are eventually zero. Thus higher-order difference domains naturally contain sequences with controlled polynomial growth. This observation is frequently useful when the original problem involves cumulative sums or discrete antiderivatives.

4.2. Growth of the inverse triangle and equality of underlying sets

The isometric identification $X(D) \cong X$ as Banach spaces does not, by itself, determine whether the two spaces consist of the same scalar sequences. That set-theoretic question is governed by the boundedness of the inverse triangle V on the model space. The explicit formula (3.3) makes the relevant threshold transparent. Put $\rho = \beta/\alpha$. The j$^{\text{th}}$ subdiagonal of V has magnitude $|\alpha|^{-m}|\rho|^j\ C(m+j-1, j)$. Thus V is a one-sided convolution operator with a negative-binomial kernel, and the summability of that kernel separates the geometrically damped case $|\rho| < 1$ from the critical case $|\rho| = 1$.

Theorem 4.4. Let X be one of c_0, c, ℓ_p $(1 \leq p \leq \infty)$, and let $D = D_\alpha, \beta^{(m)}$. If $|\beta/\alpha| < 1$, then $V = D^{-1}$ defines a bounded operator on X. Consequently $X(D) = X$ as sets and the two norms $\|\cdot\|_X$ and $\|\cdot\|_{X(D)}$ are equivalent.

Proof. Write

$h_j = \alpha^{-m}(-\beta/\alpha)^j C(m+j-1, j), j \geq 0$. The generating function of the absolute majorant is $\sum_{j\geq 0}|hj| = |\alpha|^{-m}(1 - |\beta/\alpha|)^m < \infty$. Hence V is convolution with an ℓ_1-kernel on the half-line. Young's inequality yields $\|Vy\|_p \leq \|h\|_1\|y\|_p$ for $1 \leq p \leq \infty$. The same estimate gives boundedness on c_0, because convolution with an ℓ_1 kernel preserves null sequences, and on c, because the limit of Vy is the limit of y multiplied by $\sum h_j$. Conversely D is a finite linear combination of shifts and is therefore bounded on each of these spaces. If $x \in X(D)$, then $Dx \in X$ and $x = V(Dx) \in X$; the reverse inclusion follows from boundedness of D. The norm equivalence follows from $\|Dx\|_X \leq \|D\|\|x\|_X$ and $\|x\|_X \leq \|V\|\|Dx\|_X$.

The theorem explains why a generalized difference domain can be geometrically nontrivial even when its underlying set is not enlarged. For $|\beta/\alpha| < 1$ the defining triangle merely introduces an equivalent coordinate norm on the classical space. At $|\beta/\alpha| = 1$ the negative-binomial coefficients no longer form an ℓ_1 kernel, and the inverse can create polynomially growing sequences. In particular, the classical difference choice $\beta = -\alpha$ lies precisely at this critical boundary. This distinction is useful when comparing results across the literature: two matrix domains may be isomorphic for formal reasons while displaying very different inclusion relations inside ω.

Corollary 4.5. If $|\beta/\alpha| < 1$ and $X \in \{c_0, c, \ell_p, \ell_\infty\}$, then the identity map $I: X \to X(D)$ is a topological isomorphism. More precisely, one may take

$\|x\|_{X(D)} \leq (|\alpha| + |\beta|)^m \|x\|_X$ and

$\|x\|_X \leq |\alpha|^{-m}(1 - |\beta/\alpha|)^{-m} \|x\|_{X(D)}$.

The displayed constants are not asserted to be optimal in every model space, but they are explicit and uniform in the sequence length. They also show how the norm equivalence deteriorates as $|\beta/\alpha|$ approaches one from below. This loss of uniform control anticipates the strict enlargement at the classical difference boundary.

4.3. Polynomial modes in the critical difference case

The critical case has an elementary discrete-calculus interpretation. Suppose $\beta = -\alpha$, so $D = \alpha^m (I - S)^m = \alpha^m \Delta^m$. If P is a scalar polynomial of degree $r < m$ and $x_n = P(n)$, then the ordinary mth forward/backward finite difference of P vanishes after the initial boundary terms. Consequently $\Delta^m x$ is finitely supported. Thus every polynomial sequence of degree less than m belongs to $c_0(\Delta^m)$ and to $\ell_p(\Delta^m)$ for every $1 \leq p < \infty$, even though a nonzero polynomial sequence belongs to none of those model spaces.

Proposition 4.6. Let $1 \leq p < \infty$ and $m \geq 1$. The quotient vector space $\ell_p(\Delta^m)/\ell p$ contains the m-dimensional subspace generated by the cosets of $1, n, \ldots, n^{m-1}$. The same assertion holds with c_0 in place of ℓ_p.

Proof. Membership of the polynomial sequences in the domain follows from the preceding observation. To prove linear independence modulo the model space, assume that $P(n) = \sum_{j=0}^{m-1} c_j\, n^j$ belongs to ℓ_p or c_0. Every such sequence tends to zero; a polynomial tending to zero on the nonnegative integers must be identically zero. Hence all coefficients c_j vanish.

This simple result makes precise the "discrete antiderivative" content of a difference domain. Passing from $y = \Delta^m x$ to x amounts to summing m times, and the m polynomial modes are the homogeneous solutions created by discrete integration. The boundary convention fixes a particular inverse triangle, but the large-scale polynomial behavior remains the same.

5. Schauder bases and coordinate representation

The inverse triangle provides basis vectors in a canonical way. Let $b^{(k)} = Ve^{(k)}$. By (3.3), the kth basis vector has coordinates $b_n^{(k)} = 0$ for $n < k$ and

$$b_n^{(k)} = \alpha^{-m} \left(-\frac{\beta}{\alpha}\right)^{n-k} \binom{m+n-k-1}{n-k}, \quad n \geq k. \tag{5.1}$$

Theorem 5.1. Suppose X is a BK space with AK. Then $\{b^{(k)}\}_{k \geq 0}$ is a Schauder basis for $X(D)$. If $y = Dx$, then every $x \in X(D)$ has the unique expansion $x = \sum_{k=0}^{\infty} y_k b^{(k)}$, with convergence in $\|\cdot\|_{X(D)}$.

Proof. Since X has AK, $y = \sum_{k=0}^{\infty} y_k e^{(k)}$ in X. Applying the bounded inverse isometry $V = T_D{}^{-1}$ gives $x = Vy = \sum_{k=0}^{\infty} y_k V e^{(k)}$. Norm convergence follows from $\|V(y - y^{[n]})\|_{X(D)} = \|y - y^{[n]}\|_X$. Uniqueness follows from applying D to any two such expansions.

Consequently, $c_0(D)$ and $\ell_p(D)$, $1 \leq p < \infty$, possess the explicit basis (5.1). The coefficient functionals are simply $x \mapsto (Dx)_k$. This is computationally preferable to deriving a basis by solving a separate triangular system for every k.

For $c(D)$, one may use the standard decomposition of c into the constant component and its c_0 part. If $y = Dx$ converges to L, then $y = Le + \sum_{k=0}^{\infty} (y_k - L) e^{(k)}$ in the usual basis representation of c. Applying V gives a corresponding representation of x in terms of Ve and the vectors $b^{(k)}$. The formula is useful, but the c_0 and ℓ_p cases are cleaner because the canonical finite sections already converge in norm.

5.1. Biorthogonal coordinates and the convergent model space

The basis theorem for AK spaces has a useful refinement. Let $b^{(k)} = Ve^{(k)}$ denote the k$^{\text{th}}$ inverse column. Since $Db^{(k)} = e^{(k)}$, the coefficient functional associated with $b^{(k)}$ is simply $x \mapsto (Dx)_k$. The system $\{b^{(k)}\}$ is therefore biorthogonal to the coordinate functionals induced by D. This avoids the need to compute basis coefficients by solving a new triangular system for each x.

Proposition 5.2. If $X = c_0$ or ℓ_p, $1 \le p < \infty$, then every $x \in X(D)$ has the unique expansion $x = \sum_{k=0}^{\infty} (Dx)_k b^{(k)}$, with convergence in $\|.\|_{X(D)}$. The nth partial-sum projection $P_n^D\ x = \sum_{k=0}^{n} (Dx)_k b^{(k)}$ satisfies $\|P_n^D\| = 1$.

Proof. Under the isometry T_D, P_n^D is conjugate to the canonical coordinate projection P_n on X: $P_n^D = VP_nD$. Canonical projections have norm one on c_0 and ℓ_p, and $P_n y \to y$ because these spaces have AK. Conjugation by the domain isometry gives both the norm equality and convergence.

For $c(D)$, the constant limit direction must be separated. Define $z = Ve$, where $e = (1,1, \dots)$. If $y = Dx \in c$ and $\ell = \lim y_k$, then $y = \ell e + \sum_{k=0}^{\infty} (yk - \ell) e^{(k)}$ in the standard Schauder decomposition of c. Applying V gives

$$x = \ell z + \sum_{k=0}^{\infty} ((Dx)_k - \ell)\, b^{(k)}. \tag{5.2}$$

Thus $\{z, b^{(0)}, b^{(1)}, \dots\}$ is a Schauder basis of $c(D)$. The formula is useful in applications because the distinguished vector z solves $Dz = e$ explicitly. For the first difference, $z_n = n + 1$; for higher differences, z is a polynomially growing discrete integral of the constant sequence.

5.2. Coordinate sections and approximation inside the domain

It is important to distinguish the transformed partial sums P_n^D from ordinary truncation of the original sequence x. The ordinary truncation $x^{[n]} = (x_0, \dots, x_n, 0,0, \dots)$ need not converge to x in the domain norm, even when X has AK, because applying D to $x^{[n]}$ creates an artificial boundary layer near n. By contrast, P_n^D truncates the transformed sequence Dx and then reconstructs through V, so it is adapted to the geometry of the matrix domain.

This observation yields a practical approximation principle. Given $x \in \ell_p(D)$ or $c_0(D)$, one can approximate x by vectors in $\operatorname{span}\{b^{(0)}, \dots, b^{(n)}\}$ with error exactly $\|(I - P_n)Dx\|_X$. Hence estimates that are familiar on the model space become immediate quantitative estimates in the domain. In particular, if Dx has a known tail rate, then the basis approximation in $X(D)$ has the same rate without any additional factor.

6. Dual transfer formulas

The duals of a matrix domain can be expressed by transferring the scalar series through V. Fix $a = (a_k) \in \omega$ and write $x = Vy$ with $y = Dx$. For $N \ge 0$, the N$^{\text{th}}$ partial sum of $\sum_{k=0}^{N} a_k\, x_k$ becomes

$$\sum_{k=0}^{N} a_k\, x_k = \sum_{j=0}^{N} h_{Nj}\,(a) y_j, \qquad h_{Nj}(a) = \sum_{k=j}^{N} a_k\, v_{kj}. \tag{6.1}$$

Let $H(a) = (h_{N_j}(a))_{N,j\ge 0}$, with $h_{N_j}(a) = 0$ for $j > N$. The following characterization is a direct consequence of (6.1).

Theorem 6.1. Let X be a sequence space and D a triangle with inverse V. Then $a \in (X(D))^{\beta}$ if and only if $H(a) \in (X, c)$. Similarly, $a \in (X(D))^{\gamma}$ if and only if $H(a) \in (X, \ell_\infty)$.

Proof. For $x \in X(D)$, put $y = Dx \in X$. Equation (6.1) shows that the sequence of partial sums of $\sum_{k=0}^{N} a_k\, x_k$ is exactly $H(a)y$. Hence convergence for every x is equivalent to $H(a)y \in c$ for every $y \in X$; boundedness of partial sums is equivalent to $H(a)y \in \ell_\infty$ for every $y \in X$.

For the present generalized difference

domains, v_{kj} is given explicitly by (3.3), so $H(a)$ has the computable entries

$$h_{Nj}(a) = \alpha^{-m} \sum_{k=j}^{N} a_k \left(-\frac{\beta}{\alpha}\right)^{k-j} \binom{m+k-j-1}{k-j}. \quad (6.2)$$

Thus the $\beta-$ and $\gamma-$duals are reduced to standard matrix classes on X. For $X = \ell_p$, c_0 or c, one can then insert the familiar characterizations of (X, c) and (X, ℓ_∞). The point of Theorem 6.1 is that the difference order m and the weights α, β occur only through the inverse coefficients; the target-space conditions remain unchanged.

6.1. Köthe duals through explicit auxiliary matrices

The continuous dual of $X(D)$ is abstractly identified with X^* by the isometry D, but sequence-space analysis often asks for the $\alpha-$, β-, and $\gamma-$duals, which are defined through coordinatewise products or series. These Köthe-type duals are not obtained merely by writing V^* because convergence of the defining series must be encoded. The inverse triangle nevertheless gives a clean matrix reduction.

For a scalar sequence $a = (a_n)$, define $C(a) = (c_{nk})$ by $c_{nk} = a_n v_{nk}$ for $k \le n$ and $c_{nk} = 0$ for $k > n$. Also define the partial-sum matrix $H(a) = (h_{Nk})$ by $h_{Nk} = \sum_{n=k}^{N} a_n v_{nk}$. Then for $x = Vy$ with $y \in X$,

$$(a_n x_n)_n = C(a)y,$$
$$\sum_{n=0}^{N} a_n x_n = \sum_{k=0}^{N} h_{Nk} y_k. \quad (6.3)$$

Theorem 6.2. Let X be a BK space and D an invertible triangle with inverse V. Then

(i) $a \in [X(D)]^\alpha$ if and only if $C(a) \in (X, \ell_1)$;

(ii) $a \in [X(D)]^\beta$ if and only if $H(a) \in (X, c)$; and

(iii) $a \in [X(D)]^\gamma$ if and only if $H(a) \in (X, \ell_\infty)$.

Proof. Each assertion is a direct substitution $x = Vy$. For the $\alpha-$dual, absolute summability of $(a_n x_n)$ for every $x \in X(D)$ is exactly the requirement $C(a)y \in \ell_1$ for every $y \in X$. For the $\beta-$ and $\gamma-$duals, the N[th] partial sum of $\sum a_n x_n$ is the N[th] coordinate of $H(a)y$; convergence for every y is equivalent to $H(a)y \in c$, while boundedness of all partial sums is equivalent to $H(a)y \in \ell_\infty$.

The theorem packages many lengthy dual computations into standard matrix classes. Once a catalogue of conditions for (X, ℓ_1), (X, c), and (X, ℓ_∞) is available, the duals follow by substituting the explicit negative-binomial coefficients v_{nk}. This is especially effective for $X = \ell_p$, where row norms and classical matrix criteria reduce the problem to weighted sums involving $C(m+n-k-1, n-k)$.

Example 6.3. For the first difference Δ, $v_{nk} = 1$ for $k \le n$. Hence $C(a)$ has entries $c_{nk} = a_n$ on the lower triangle, while $h_{Nk} = \sum_{n=k}^{N} a_n$. The $\beta-$dual condition is therefore expressed through the matrix of tails of the coefficient sequence. Higher-order differences replace these simple tails by binomially weighted tails, precisely as repeated discrete summation suggests.

7. Infinite matrix transformations from the domain

Let $B = (b_{nk})$ be an infinite matrix. To transfer B from $X(D)$ to X, formally substitute $x = Vy$. If the relevant series converge, then

$$(Bx)_n = \sum_{k=0}^{\infty} b_{nk} x_k = \sum_{j=0}^{\infty} \left[\sum_{k=j}^{\infty} b_{nk} v_{kj} \right] y_j. (7.1)$$

Define the associated matrix $\hat{B} = (\hat{b}_{nj})$ by

$$\hat{b}_{nj} = \sum_{k=j}^{\infty} b_{nk} v_{kj}. \quad (7.2)$$

whenever the series exists. In operator notation, $\hat{B} = BV$. The transformed matrix is the correct object for norms and mapping conditions because $B = \hat{B}D$ on the domain.

Theorem 7.1. Let X and Y be sequence spaces, D a triangle with inverse V, and B an infinite matrix. Assume each row B_n belongs to $(X(D))^\beta$ so that Bx is well defined on $X(D)$. Then $B \in (X(D), Y)$ if and only if the associated matrix $\hat{B} = BV$ belongs to (X, Y).

Proof. For every $x \in X(D)$, set $y = Dx$. By the row condition and the triangular expansion, (7.1) gives $Bx = \hat{B}y$. Since T_D maps $X(D)$ onto X, the image $B(X(D))$ is contained in Y exactly when $\hat{B}(X)$ is contained in Y.

If X and Y are BK spaces and the matrix maps are bounded, Theorem 7.1 can be read as an operator factorization $L_B = L_{\hat{B}} T_D$. Because T_D is an isometry, the operator norm of B on the domain equals the norm of the associated operator on X whenever Y is equipped with its given norm: $\|L_B\| = \|L_{\hat{B}}\|$. This identity is the fundamental reason that matrix-domain operator problems can often be solved without estimating the original rows b_n directly.

7.1. Example: first-order weighted differences

Take $m = 1$ and $D = I - \theta S$ with $|\theta| < 1$. Then $v_{nk} = \theta^{n-k}$. For a matrix B on $\ell_p(D)$, the associated entries are $\hat{b}_{nj} = \sum_{\mathrm{k=j}}^{\infty} b_{nk}\theta^{k-j}$. Thus each row is transformed by a one-sided geometric averaging. In the special case that B is diagonal, $b_{nk} = \gamma_n \delta_{nk}$, one obtains $\hat{b}_{nj} = \gamma_n \theta^{n-j}$ for $j \le n$ and zero otherwise. This explicit row form will reappear in compactness criteria based on dual norms. Related compactness analyses for difference sequence spaces use the same associated-row philosophy [11].

7.2. Bounded matrix maps from $\ell_p(D)$ into classical targets

The associated-matrix method becomes particularly concrete for ℓ_p domains. Let $1 \le p < \infty$, let q be conjugate to p when $p > 1$ and put $q = \infty$ when $p = 1$. If $B = (b_{nk})$ acts on $\ell_p(D)$, define the associated rows by $\hat{b}_{nj} = \sum_{\mathrm{k=j}}^{\infty} b_{nj} v_{jk}$, whenever the sums converge and the interchange of summation is justified. Then $Bx = \hat{B}(Dx)$, so boundedness can be read on the ordinary ℓ_p space.

Theorem 7.2. Assume the associated matrix $\hat{\mathrm{B}}$ is well defined. Then $B \in (\ell_p(D), \ell_\infty)$ if and only if $\sup_n \| \hat{b}_n \|_q < \infty$. In that case

$$\| B \|_{\ell_p(D) \to \ell_\infty} = \sup_n \| \hat{b}_n \|_q. \qquad (7.3)$$

If, in addition, $\hat{b}_{nk} \to 0$ as $n \to \infty$ for every fixed k, then B maps $\ell_p(D)$ into c_0. Conversely, boundedness into c_0 implies these coordinatewise limits. For $1 < p < \infty$ this follows from the standard characterization of (ℓ_p, c_0); for $p = 1$ the same statement uses the ℓ_∞ norm of the rows.

Proof. The equality $B = \hat{B}D$ and the isometry $D: \ell_p(D), \ell_p$ reduce the first claim to the norm of a matrix map from ℓ_p to ℓ_∞. Hölder's inequality gives $|(\hat{B}y)_n| \le \|\hat{b}_n\|_q \|y\|_p$, and taking a near norming vector for a row yields equality after the supremum. The c_0 statement follows by testing against finitely supported vectors for necessity and approximating arbitrary $y \in \ell_p$ by finite sections for sufficiency.

A similar criterion describes maps into c: one requires coordinatewise convergence $\hat{b}_{nk} \to a_k$ together with uniform row boundedness, and the limiting row a must define a continuous functional on ℓ_p. The limit of $(Bx)_n$ is then $\sum_k a_k (Dx)_k$. Thus even the asymptotic output of B is most naturally expressed in transformed coordinates.

7.3. Endomorphisms and the associated operator algebra

When B maps $X(D)$ into itself, multiplication only on the right by V is not sufficient because the output norm also contains D. The correct model operator is the full conjugate $C = DBV$. At the abstract level this gives an isometric algebra identification between bounded operators on $X(D)$ and bounded operators on X.

Theorem 7.3. The map $\Phi: B(X(D)) \to B(X)$, $\Phi(B) = DBD^{-1}$, is an isometric unital algebra isomorphism. It preserves rank, finite rank, compactness, invertibility and algebraic identities. In particular, if P is a polynomial, then $\Phi(P(B)) = P(\Phi(B))$.

Proof. The identity follows from conjugation by the isometric isomorphism D:X(D)→X. Multiplicativity is $\Phi(B_1B_2) = DB_1D^{-1}DB_2D^{-1} = \Phi(B_1)\Phi(B_2)$, and the norm equality follows because both D and D^{-1}, regarded between the domain and model spaces, have norm one. Rank and compactness are preserved by composition with bounded isomorphisms, and the remaining assertions follow algebraically.

The theorem places the matrix-domain construction in a broader operator-theoretic perspective. Although the coordinate matrix of B on $X(D)$ may be complicated, the operator algebra is merely a conjugate copy of $B(X)$. This observation is the natural gateway to spectral theory: resolvents, spectra and Fredholm properties of B are those of DBD^{-1}. It also underlies compactness criteria, where the relevant tails belong to the associated model operator rather than to the raw coefficients of B.

7.4. Worked second-order example

Take $m = 2$, $\alpha = 1$ and $\beta = -1$. Then $D = \Delta^2$ and $(Dx)_n = x_n - 2x_{n-1} + x_{n-2}$, with the convention that negative-index terms are zero. The inverse coefficients are $v_{nk} = n - k + 1$ for $k \leq n$, so

$$x_n = \sum_{k=0}^{n} (n - k + 1)\, y_k, \qquad y = \Delta^2 x. \quad (7.4)$$

The inverse columns are therefore $b^{(k)} = (0, \ldots, 0,1,2,3, \ldots)$ beginning at index k. If $y \in \ell_p$, formula (7.4) produces an element of $\ell_p(\Delta^2)$ that may grow linearly. For a matrix B acting from $\ell_p(\Delta^2)$ to ℓ_∞, the associated coefficient becomes $\hat{b}_{nk} = \sum_{\mathrm{j=k}}^{\infty} (\mathrm{j} - \mathrm{k} + 1) b_{nj}$. Thus boundedness is controlled not by the ordinary row b_n but by its first moment-like tails. This explicit example illustrates the general theme: the order of the difference determines the polynomial weight appearing in the transformed rows.

For the damped operator $D = (I - \theta S)^2$ with $|\theta| < 1$, the inverse weight $(j - k + 1)$ is multiplied by θ^{j-k}. The resulting kernel is summable, so the domain collapses set-theoretically to the model space and the moment-like tail is exponentially regularized. The comparison between $\theta = 1$ and $|\theta| < 1$ provides a concrete way to see the phase change described in Theorem 4.4.

7.5. Nested domains generated by increasing difference order

The family $D^{(m)} = (\alpha I + \beta S)^m$ also has an internal semigroup structure: $D^{(m+r)} = D^{(r)} D^{(m)}$. Since every finite-band polynomial in S is bounded on c_0, c, ℓ_p and ℓ_∞, this identity produces natural inclusions between domains of different orders. The effect is clearest for the classical choice $\Delta = I - S$.

Theorem 7.4. Let $X \in \{c_0, c, \ell_p, \ell_\infty\}$ and $m, r \geq 1$. Then $X(\Delta^m) \subset X(\Delta^{m+r})$. If $X = c_0$ or ℓ_p with $1 \leq p < \infty$, the inclusion $X(\Delta^m) \subset X(\Delta^{m+1})$ is strict for every m.

Proof. If $x \in X(\Delta^m)$, then $y = X(\Delta^m x) \in X$. Since Δ^r is bounded on X, $\Delta^{m+r} x = \Delta^r y \in X$, proving the inclusion. For strictness, take $x_n = n^m$. Its (m+1)^st difference is eventually zero, so $x \in X(\Delta^{m+1})$; but its m^th difference is eventually a nonzero constant, which belongs neither to c_0 nor to ℓ_p. Hence $x \notin X(\Delta^m)$.

Thus the classical difference domains form a strictly increasing scale

$$\begin{gathered} X \subsetneq X(\Delta) \subsetneq X(\Delta^2) \subsetneq \cdots, \\ X = c_0 \text{ or } \ell_p, \quad 1 \leq p < \infty. \end{gathered} \quad (7.5)$$

Each step allows one additional polynomial degree of asymptotic growth. This hierarchy is a discrete analogue of Sobolev-type scales defined through derivatives, except that the domain condition is imposed on a difference rather than on the sequence itself. The analogy should be used cautiously—the spaces remain isomorphic to X under their graph norms—but it is useful for organizing discrete regularity and antiderivative behavior.

For the damped case $|\beta/\alpha| < 1$, Theorem 4.4 shows that this set-theoretic hierarchy disappears: every $X(D^{(m)})$ coincides with X, although the graph norms change with m. Hence strict nesting is tied to the loss of ℓ_1 summability of the inverse kernel at the critical ratio.

7.6. Norms of coordinate-series functionals

A functional on $X(D)$ is often presented as a coordinate series $f_a(x) = \sum_{n \geq 0} a_n x_n$. When this series represents a continuous functional, the inverse triangle identifies its effective coefficient sequence on X. Suppose that the rearrangement is justified and define

$$\tilde{a}_k = \sum_{n=k}^{\infty} a_n v_{nk}. \tag{7.6}$$

Then $f_a(Vy) = \sum_k \tilde{a}_k y_k$. For $X = \ell_p$, $1 < p < \infty$, this yields an exact dual norm whenever $\tilde{a} \in \ell_q$; for $X = c_0$, it yields the ℓ_1 norm.

Proposition 7.5. Let $1 < p < \infty$ and $q = p/(p-1)$. If a is such that (7.6) converges and f_a is represented on $\ell_p(D)$ by the transformed coefficient sequence $\tilde{a} \in \ell_q$, then

$$\| f_a \|_{(\ell_p(D))^*} = \| \tilde{a} \|_q. \tag{7.7}$$

For $c_0(D)$, the corresponding formula is $\|f_a\| = \sum_k |\tilde{a}_k|$. For $\ell_1(D)$, the dual coefficient lies in ℓ_∞ and the norm is $\sup_k |\tilde{a}_k||$.

Proof. Under the isometric isomorphism D, f_a corresponds to the functional $y \mapsto \sum \tilde{a}_k y_k$ on the model space. The standard duality identifications $(\ell_p)^* = \ell_q$, $(c_0)^* = \ell_1$ and $((\ell_1)^* = \ell_\infty$ give the formulas. The substantive sequence-space work lies in verifying that the original coefficient sequence a indeed permits the transformation and rearrangement; this is exactly what the β −dual matrix $H(a)$ controls.

For Δ^m the transformed coefficient is a binomially weighted tail because $v_{nk} = C(m+n-k-1, n-k)$. Thus a coordinate functional that looks local in x can become a long-range functional in $y = \Delta^m x$. This dual effect mirrors the row transformation for matrix operators.

7.7. A summation operator as the inverse of the first difference

The first-difference domain gives a useful sanity check for all formulas. Let $D = \Delta$. Then V is the summation triangle, $(Vy)_n = \sum_{k=0}^{n} y_k$. On $c_0(\Delta)$, the basis vector $b^{(k)}$ is the step sequence that is zero before k and one from k onward. A vector x has the expansion

$$x = \sum_{k=0}^{\infty} (x_k - x_{k-1})\, b^{(k)}, \qquad x_{-1} = 0. \tag{7.8}$$

with convergence in the difference norm. Although the basis vectors do not belong to c_0 as ordinary sequences, they are perfectly legitimate elements of $c_0(\Delta)$ because their differences are coordinate vectors. This example emphasizes why properties of the domain should be assessed in the graph norm rather than from pointwise intuition.

If B is a matrix on $c_0(\Delta)$, its associated row is $\hat{b}_{nk} = \sum_{j=k}^{\infty} b_{nj}$. Thus the natural coefficients are row tails. For Δ^2 they are first moments of row tails, and for Δ^m they are polynomially weighted tails of order m-1. The progression makes the negative-binomial inverse formula operational rather than merely formal.

7.8. Stability with respect to the parameters α and β

The family $D_\alpha, \beta^{(m)}$ depends polynomially on α and β at the level of the forward triangle, while its inverse depends analytically on the ratio β/α as long as $\alpha \neq 0$. This permits a simple perturbation comparison between nearby generalized difference domains. Fix m and suppose α and α' are nonzero. On each classical model space X, the finite-band operators $D_\alpha, \beta^{(m)}$ and $D_{\alpha'}, \beta'^{(m)}$ satisfy an operator-norm estimate controlled by the differences of their binomial coefficients. Therefore the graph norms vary continuously with the parameters whenever the inverse operators remain uniformly bounded on X.

In particular, on any compact parameter set K contained in $\{(\alpha, \beta): \alpha \neq 0, |\beta/\alpha| < 1\}$, the kernels of the inverses have uniformly bounded ℓ_1 norms. Hence the equivalence constants in Corollary 4.5 can be chosen uniformly for $(\alpha, \beta) \in K$. The matrix domains then form a stable family of equivalent realizations of X. As the ratio approaches the unit circle, this uniform bound can fail, reflecting the emergence of polynomially growing inverse coefficients.

This parameter stability is useful when a difference model contains estimated or uncertain coefficients. Small changes in α and β away from the critical boundary alter the domain norm continuously rather than changing the underlying set abruptly. The qualitative transition occurs only when the inverse kernel loses absolute summability, which explains why the ratio

$|\beta/\alpha| = 1$ is the natural threshold in the present family.

A similar observation applies to finite products of generalized difference triangles. If $A = D_{\alpha_1}, \beta_1^{(m_1)} \cdots D_{\alpha_r}, \beta_r^{(m_r)}$ and every $|\beta j/\alpha j| < 1$, then $A \; and \; A^{-1}$ are bounded on the classical model spaces, so $X(A) = X$ with an equivalent norm. If one factor is critical, the product domain can acquire the polynomial modes generated by that factor. Thus the inverse factorization provides a direct diagnostic for whether a composite matrix domain genuinely enlarges the model space as a subset of ω.

7.9. Consequences for discrete regularity problems

The preceding structure suggests a convenient way to formulate discrete regularity assumptions. Instead of requiring a sequence x itself to be summable or convergent, one may require $D_\alpha, \beta^{(m)}x$ to lie in a chosen model space. The transformed sequence then acts as a discrete derivative, while the inverse triangle reconstructs x by a weighted discrete integration. The explicit coefficients identify exactly how local transformed data propagate into the original coordinates.

For the classical operator Δ^m, boundedness of the transformed sequence controls the m[th] discrete derivative but leaves polynomial modes of degree below m unconstrained. For $|\beta/\alpha| < 1$, those modes are exponentially damped by the inverse and no set-theoretic enlargement occurs. Hence the parameter ratio distinguishes a genuinely integrative regime from a stable filtering regime.

This viewpoint can be useful in difference equations. If a forcing sequence y belongs to X and one solves $D_\alpha, \beta^{(m)}x = y$, the unique solution selected by the triangular boundary convention is $x = Vy$. Membership and norm estimates are immediate from the model-space properties of y. Additional linear conditions can then be imposed on the finitely many low-order modes when an alternative boundary normalization is desired.

More broadly, the matrix-domain method separates three issues that are often mixed together: the algebra of the difference operator, the geometry of the target model space, and the coordinate growth of the inverse. Once these are treated separately, basis, duality and matrix-transformation results become consequences of a small number of transfer identities rather than independent calculations for each newly defined sequence space.

8. Discussion and concluding remarks

The family $D_\alpha, \beta^{(m)}$ offers a convenient algebraic model for generalized difference domains because it combines three useful features: it is a finite-band triangle, its inverse is explicit, and its order is controlled by a single integer m. The matrix-domain construction then separates the coordinate description from the Banach-space geometry. On the one hand, $X(D)$ can contain sequences that are far outside X; on the other hand, the transform D identifies the domain isometrically with X. This dual perspective clarifies why many apparently new difference spaces inherit familiar structural properties while still producing nontrivial matrix transformations.

The explicit inverse gives more than a completeness proof. It produces the Schauder basis, converts scalar series into the matrix $H(a)$, and converts an operator B on the domain into the associated matrix BV on the model space. These transformations make the generalized difference parameters visible in a controlled way and avoid repeated ad hoc elimination of the coordinates x_k.

Two natural continuations follow. The first is spectral: if an operator acts from a matrix domain into itself, conjugation by the defining triangle transfers the resolvent and spectrum to the model space. The second is compactness: because the triangle is an isometry under the natural domain norm, the Hausdorff measure of non-compactness can also be transported to an associated operator on the model space. Those two directions provide natural operator-theoretic continuations of the matrix-domain framework developed here. This transfer is consistent with Hausdorff-measure-of-noncompactness approaches to compact matrix operators on BK and difference sequence spaces [7,8,11].

9. Further extensions and research directions

The polynomial family $(\alpha I + \beta S)^m$ is only one convenient realization of a generalized difference triangle. The same arguments work for any finite product $D = (\alpha_1 I + \beta_1 S) \cdots (\alpha_m I + \beta_m S)$ with every α_j nonzero. In that case the inverse is the product of the corresponding geometric inverse triangles in reverse order. Although a single closed coefficient formula may be less compact, the isometric-domain theorem, basis construction and matrix-transfer result remain unchanged. This permits nonhomogeneous difference weights to be introduced without rebuilding the functional-analytic theory.

A second direction is to replace the integer order by a fractional difference triangle defined through generalized binomial coefficients. Provided the diagonal stays nonzero and the inverse triangle is well defined, the matrix-domain mechanism again survives. The principal new issue is no longer completeness but the asymptotic behavior of the inverse coefficients, which determines inclusion relations and the strength of compactness conditions for associated operators. In this sense, the inverse triangle is the true analytic datum of a generalized difference domain.

A third direction concerns interpolation and geometry. Since $\ell_p(D)$ is isometric to ℓ_p under the natural norm, uniform convexity, type, cotype and reflexivity are inherited for $1 < p < \infty$. However, coordinatewise order properties and solidity may fail because the norm is generally non-absolute. It is therefore useful to distinguish Banach-space properties, which are transported by the isometry, from lattice or coordinate properties, which depend on the entries of D and V. This distinction can prevent incorrect conclusions when new matrix domains are compared only by their defining coordinate formulas.

Finally, the associated-matrix formula BV suggests a practical workflow for future investigations: first compute or estimate the inverse triangle V; second transfer the proposed matrix operator to the model space; third use established criteria on c_0, c or ℓ_p; and only then translate the conclusion back to the original coordinates. This workflow is especially effective for spectral, Fredholm and compactness problems because those properties are stable under the isomorphisms generated by triangles.

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