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Solvability and Spectrum of Infinite Matrix Operators on XA

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Abstract **Original Research Article**

Let X be a Banach sequence space, let A be an invertible triangle, and equip the matrix domain $X_A = \{x: Ax \in X\}$ with the norm $\|x\|_A \|Ax\|_X$ This paper studies solvability and spectral properties of infinite matrix operators acting on X_A. The central observation is that the coordinate map $U_A: X_A \to X$, $U_A x = Ax$, is an isometric isomorphism, so every bounded operator B on X_A is similar to $C = U_A B U_A{}^{-1}$ on X. The similarity yields exact transfer principles for the resolvent, spectrum, point spectrum, approximate point spectrum, continuous and residual spectra, Fredholmness, index and essential spectrum. A corresponding solvability formula converts $(B - \lambda I)x = f$ on X_A into $(C - \lambda I)y = Af$ on X. When B and A are given by infinite matrices, explicit formulas for the associated conjugate matrix are obtained under standard row-convergence hypotheses. Diagonal and shift models illustrate how nontrivial operators on difference-type matrix domains can have spectra computed without direct coordinate recursion. Perturbation, pseudospectral, compact-perturbation and Neumann-series criteria are also recorded. The framework provides a general mechanism for transporting spectral questions from a matrix domain to a familiar model space.

Keywords: matrix domain; infinite matrix; spectrum; resolvent; solvability; Fredholm operator; similarity.

1. Introduction

Spectral analysis of infinite matrices on classical sequence spaces has a long history, and difference matrices have been a particularly active source of examples. Fine spectra of two-band and three-band generalized difference operators have been determined on c_0, c, ℓ_p and bounded-variation spaces, while later work refined the spectrum into point, continuous, residual and other subdivisions [1-4]. Parallel to this spectral literature, matrix-domain methods have created many new sequence spaces by imposing a triangular transform on a classical space [5-7]. These two lines of work naturally meet when an infinite matrix acts not on X itself but on a matrix domain X_A. Representative studies include Altay and Başar [1], Bilgiç and Furkan [2], Başar, Durna and Yıldırım [3], Furkan, Bilgiç and Başar [4], Başar and Karaisa [8], and Alotaibi, Yaying and Mohiuddine [9].

A direct spectral calculation on X_A can be cumbersome. The defining norm already contains A, while the operator B contributes another matrix. Solving $(B - \lambda I)x = f$ by coordinates may therefore lead to nested recursions. The essential simplification is to use A as a change of coordinates. When A is a triangle and X_A carries the natural norm $\|x\|_A = \|Ax\|_X$, the map $U_A x = Ax$ is an isometric isomorphism from X_A onto X. Consequently, B on X_A is similar to $C = U_A B U_A{}^{-1}$ on X. Since similarity preserves invertibility and the main spectral classes, the entire solvability problem can be transported to X.

This observation is elementary at the level of abstract operator theory, but it is highly effective in the matrix-domain setting because the similarity can be written explicitly in terms of infinite matrices. If $V = A^{-1}$, then the conjugated operator is represented formally by $C = ABV$. For triangular, banded, or sufficiently regular matrices the entries of C are computable. The transformed equation $(C - \lambda I)y = Af$ has the same condition number in the isometric coordinate system and its solution gives $x = Vy$. Thus spectral and solvability statements can be established on the model space and then pulled back to the domain.

The paper has four main aims. First, it formulates the similarity theorem for bounded operators on X_A and gives a precise matrix realization. Second, it establishes exact resolvent and solvability transfer formulas. Third, it records the invariance of spectral subdivisions, Fredholm properties and the essential spectrum. Fourth, it demonstrates the method with diagonal and shift models and with perturbation criteria. The results are deliberately stated for a general Banach sequence space X so that difference domains, weighted-mean domains and other triangular constructions are covered simultaneously.

2. Matrix-domain setting and operator realization

Let X be a Banach sequence space and $A = (a_{nk})$ a triangle. Write $V = A^{-1}$ and define $X_A = \{x \in \omega : Ax \in X\}$ with $\|x\|_A = \|Ax\|_X$. As in standard matrix-domain theory [5–7], $U_A : X_A \to X$, $U_A x = Ax$, is an isometric isomorphism with inverse $U_{A^{-1}} y = Vy$. We shall use $B(E)$ for the bounded linear operators on a Banach space E.

Theorem 2.1. For every $B \in B(X_A)$, the operator $C = U_A B U_A{}^{-1}$ belongs to $B(X)$, and $B = U_A{}^{-1} C U_A$. Moreover $\|C\| = \|B\|$.

Proof. Because U_A and $U_A{}^{-1}$ are isometries, C is bounded and $\|C\| \le \|B\|$. Conversely $B = U_A{}^{-1} C U_A$ gives $\|B\| \le \|C\|$. Hence equality holds.

Theorem 2.1 says that $B(X_A)$ and $B(X)$ are isometrically isomorphic as Banach algebras through $B \mapsto U_A B U_A{}^{-1}$. In particular, algebraic identities, powers and polynomials are transferred exactly: $C^n = U_A B^n U_A{}^{-1}$ and $p(C) = U_A p(B) U_A{}^{-1}$ for every polynomial p.

2.1. Infinite-matrix form

Assume now that $B = (b_{ij})$ is represented by an infinite matrix on X_A. For $y \in X$ and $x = Vy$, one first forms $B(Vy)$, then applies A. Under convergence conditions permitting the indicated rearrangements, the conjugate matrix $C = (c_{nk})$ is

$$c_{nk} = \sum_{i=0}^{n} a_{ni} \sum_{j=k}^{\infty} b_{ij}\, v_{jk}. \qquad (2.1)$$

The outer sum is finite because A is triangular. The inner sum is exactly the associated-row transform of B by V. If B is banded or triangular, the number of nonzero terms can also be finite. Formula (2.1) should be used together with the requirement that every row of B defines a continuous coordinate functional on X_A. Once B is known to be bounded, the abstract conjugate C exists independently of whether one chooses to compute every entry by (2.1). The row-continuity and rearrangement requirements used here are consistent with the classical matrix-transformation framework summarized by Stieglitz and Tietz [10].

For generalized difference domains generated by the higher-order triangle $D_\alpha, \beta^{(m)}$, formula (2.1) converts an operator on the difference domain into a matrix on c_0, c or ℓ_p whose rows contain explicit weighted sums of the original entries. This avoids solving the difference recursion anew for each value of the spectral parameter.

3. Solvability of $(B - \lambda I)x = f$

Fix $\lambda \in \mathbb{C}$. The equation of interest is

$$(B - \lambda I)x = f, \qquad x, f \in X_A. \qquad (3.1)$$

Set $y = U_A x = Ax$ and $g = U_A f = Af$. Since $U_A I X_A U_A{}^{-1} = IX$, equation (3.1) is equivalent to $(C - \lambda I)y = g, \qquad y, g \in X,$

$$C = U_A B U_A^{-1}. \qquad (3.2)$$

Theorem 3.1 (Solvability transfer). For $B \in B(X_A)$, $C = U_A B U_A^{-1}$, $\lambda \in \mathbb{C}$ and $f \in X_A$, equation $(B - \lambda I)x = f$ has a solution $x \in X_A$ if and only if $(C - \lambda I)y = Af$ has a solution $y \in X$. Solutions correspond bijectively by x=Vy and $y = Ax$. Uniqueness is preserved.

Proof. Apply U_A to $(B - \lambda I)x = f$. Since $U_A B = C U_A$, one obtains $(C - \lambda I)Ax = Af$. Conversely, if y solves (3.2), then $x = Vy$ lies in X_A and applying U_A^{-1} to (3.2) yields (3.1). The correspondence is bijective because U_A is bijective.

Corollary 3.2. If λ belongs to the resolvent set of C, then it belongs to the resolvent set of B and

$$(B - \lambda I)^{-1} = V(C - \lambda I)^{-1}A,$$

$$\| (B - \lambda I)^{-1} \| = \| (C - \lambda I)^{-1} \|. \qquad (3.3)$$

The norm identity in (3.3) is specific to the natural matrix-domain norm, which makes U_A an isometry. If X_A were equipped with an equivalent but different norm, similarity would still preserve the spectrum, but the resolvent norms would generally be comparable rather than equal.

3.1. Neumann-series region

Proposition 3.3. If $|\lambda| > \|B\| = \|C\|$, then $\lambda \in \rho(B)$ and

$$(B - \lambda I)^{-1} = -\lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} B^n,$$

$$\| (B - \lambda I)^{-1} \| \leq \frac{1}{|\lambda| - \|B\|}. \qquad (3.4)$$

Proof. The standard Neumann series applies to $I - \lambda^{-1}B$ because $\|\lambda^{-1}B\| < 1$. The same conclusion follows after transfer to C. The equality of norms makes the exterior resolvent estimate identical in the model coordinates.

Although (3.4) is elementary, it gives an immediate solvability region for infinite matrix equations on X_A without estimating the coordinates of x. The bound can often be sharpened after conjugation because C may have a recognizable form, such as a diagonal or weighted-shift operator, for which the exact resolvent is known.

3.2. Kernel, range and quantitative stability of solutions

The similarity does more than identify the set of spectral parameters. It transports the entire geometry of the equation. Let $T_\lambda = B - \lambda I$ on X_A and $S_\lambda = C - \lambda I$ on X, where $C = U_A B U_A^{-1}$. Then $S_\lambda U_A = U_A T_\lambda$. Consequently kernels, ranges and quotient spaces correspond exactly under U_A.

Proposition 3.4. For every $\lambda \in \mathbb{C}$, $U_A(\ker T_\lambda) = \ker S_\lambda$ and $U_A\big(R(T_\lambda)\big) = R(S_\lambda)$. Hence $\dim \ker T_\lambda = \dim \ker S_\lambda$ and $\operatorname{codim} R(T_\lambda) = \operatorname{codim} R(S_\lambda)$, with the convention that the dimensions may be infinite. The range $R(T_\lambda)$ is closed if and only if $R(S_\lambda)$ is closed.

Proof. If $T_\lambda x = 0$, then $S_\lambda(U_A x) = U_A T_\lambda x = 0$; applying U_A^{-1} proves the reverse inclusion. The same intertwining identity gives $U_A(R(T_\lambda)) = R(S_\lambda)$. Since U_A is a homeomorphism, it preserves closedness and induces an isomorphism of the corresponding quotient spaces.

For well-posedness it is useful to record the minimum modulus $m(T) = \inf\{\|Tx\| : \|x\| = 1\}$. Because U_A is an isometry,

$$m(T_\lambda) = m(S_\lambda). \qquad (3.5)$$

Thus T_λ is bounded below exactly when S_λ is bounded below, and the same lower bound works in both coordinate systems. If $\lambda \in \rho(B)$, then

$$\| (B - \lambda I)^{-1} \|_{X_A \to X_A} = \| (C - \lambda I)^{-1} \|_{X \to X}. \qquad (3.6)$$

This equality has a direct numerical interpretation. The change of coordinates imposed by the matrix-domain norm introduces no additional condition-number penalty. Any growth of the solution operator near the spectrum is already present in the model problem. In particular, stable inversion regions and ill-conditioned spectral neighborhoods can be studied entirely on X.

Corollary 3.5. Suppose S_λ is bounded below by $\delta > 0$. Then for every $x \in X_A$, $\|(B - \lambda I)x\|_A \geq \delta \|x\|_A$. Hence the homogeneous equation has only the zero solution and every approximate solution obeys the a posteriori estimate $\|x\|_A \leq \delta^{-1} \|(B - \lambda I)x\|_A$.

3.3. Formal triangular recursion versus Banach-space solvability

When the conjugated matrix $C = (c_{nk})$ is triangular, one can often solve the coordinate equations recursively. If λ differs from every diagonal entry c_{nn}, the formal relation $(C - \lambda I)y = g$ determines a unique scalar sequence $y \in \omega$ by

$$y_n = (c_{nn} - \lambda)^{-1}[g_n - \textstyle\sum_{k=0}^{n-1} c_{nk}\, y_k]. \qquad (3.7)$$

However, formal recursion in ω is not the same as solvability in X. The recursively produced y must belong to the model space, and this global membership requirement is precisely where the spectrum enters. A parameter λ can avoid every diagonal entry and still belong to the spectrum because the inverse recursion fails to define a bounded operator on X. Shift operators provide the standard example: their diagonal is zero, but their spectrum contains an entire disk rather than merely $\{0\}$.

This distinction is particularly important for matrix domains. A coordinate calculation performed directly on X_A may appear to produce a solution sequence while obscuring whether the solution has finite domain norm. The conjugation method separates the algebraic recursion from the functional-analytic requirement $y \in X$ and thereby prevents a common source of erroneous resolvent claims.

4. Spectrum and spectral subdivisions

Recall that the spectrum $\sigma(T)$ of $T \in B(E)$ consists of those λ for which $T - \lambda I$ fails to be boundedly invertible; the resolvent set is $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Similarity of bounded operators preserves the spectrum. In the present setting the statement has a particularly transparent proof.

Theorem 4.1 (Spectral invariance). Let $B \in B(X_A)$ and $C = U_A B U_A^{-1}$. Then $\sigma(B) = \sigma(C)$ and $\rho(B) = \rho(C)$.

Proof. For each λ, $C - \lambda I = U_A(B - \lambda I)U_A^{-1}$. Therefore $B - \lambda I$ is invertible if and only if $C - \lambda I$ is invertible, and their inverses are related by (3.3).

More refined spectral classes are also preserved because U_A is a homeomorphism and carries kernels and ranges bijectively. Write σ_p for the point spectrum, σ_{ap} for the approximate point spectrum, σ_c for the continuous spectrum and σ_r for the residual spectrum, using the standard Banach-space definitions.

Theorem 4.2. Under the hypotheses of Theorem 4.1, the point, approximate point, continuous and residual spectra are invariant under the matrix-domain similarity: $\sigma_p(B) = \sigma_p(C)$, $\sigma_{ap}(B) = \sigma_{ap}(C)$, $\sigma_c(B) = \sigma_c(C)$, and $\sigma_r(B) = \sigma_r(C)$.

Proof. The identity $\ker(C - \lambda I) = U_A \ker(B - \lambda I)$ gives the point spectrum. If $\|x_n\|_{X_A} = 1$ and $(B - \lambda I)x_n \to 0$, then $y_n = U_A x_n$ has $\|y_n\|_X = 1$ and $(C - \lambda I)y_n \to 0$, and the converse is identical; this proves approximate-point invariance. Finally, U_A maps $R(B - \lambda I)$ onto $R(C - \lambda I)$, preserving closure, density and surjectivity, so the continuous and residual classifications coincide.

The result includes the spectral subdivisions commonly used in fine-spectrum studies of generalized difference matrices [1–4, 8, 9]. The advantage is conceptual: when the operator on X_A is obtained by transporting a familiar operator on X, no separate fine-spectrum calculation is needed. Conversely, a difficult matrix B may become a simple C after conjugation by A.

4.1. Approximate eigenvectors and spectral boundary

Recall that λ belongs to the approximate point spectrum $\sigma_{ap}(T)$ if there exists a sequence of unit vectors x_j with $\|(T - \lambda I)x_j\| \to 0$. The similarity immediately transports approximate eigenvectors: if $y_j = U_A x_j$, then $\|y_j\| = 1$ and $\|(C - \lambda I)y_j\| = \|(B - \lambda I)x_j\|_A$. Therefore, $\sigma_{ap}(B) = \sigma_{ap}(C)$.

Proposition 4.3. The boundary $\partial\sigma(B)$ is contained in $\sigma_{ap}(B)$, and under the matrix-domain isometry it corresponds to $\partial\sigma(C) \subset \sigma_{ap}(C)$. In particular, whenever a model operator admits explicit approximate eigenvectors at spectral boundary points, applying A^{-1} produces approximate eigenvectors in X_A with exactly the same defect norm.

This is useful for shift-like models. On c_0 and ℓ_p, normalized vectors supported on long blocks can

be used to approximate unimodular eigenmodes of the unilateral shift. If $B = A^{-1}SA$, the corresponding vectors $A^{-1}y_j$ may look highly nonlocal in the original coordinates, yet no new estimate is needed: the defect is unchanged by the isometry.

4.2. Polynomial and holomorphic spectral mapping

Similarity is compatible with functions of an operator. If p is a polynomial, then $p(B) = U_{A^{-1}}p(C)U_A$. The ordinary spectral mapping theorem therefore gives

$$\sigma\big(p(B)\big) = p\big(\sigma(B)\big) = p\big(\sigma(C)\big). \tag{4.1}$$

More generally, if f is holomorphic on a neighborhood of $\sigma(B)$, the holomorphic functional calculus is similarity invariant and $f(B) = U_{A^{-1}}f(C)U_A$. Consequently $\sigma(f(B)) = f(\sigma(B))$. This observation permits semigroups, resolvent powers and rational functions of matrix-domain operators to be studied through the model operator whenever the relevant functional calculus is defined.

An immediate special case concerns powers. If $B = A^{-1}CA$, then $B^n = A^{-1}C^nA$ and $\|B^n\| = \|C^n\|$. Hence the spectral-radius formula yields $r(B) = r(C)$, not merely as an abstract similarity statement but with equality of every power norm in the natural domain norm.

4.3. Pseudospectra and exact resolvent-norm transfer

For $\varepsilon > 0$ define the ε-pseudospectrum by $\sigma_\varepsilon(T) = \sigma(T) \cup \{\lambda \in \rho(T): \|(T - \lambda I)^{-1}\| > \varepsilon^{-1}\}$. Non-unitary similarities on Hilbert or Banach spaces can distort pseudospectra severely because the norms of the intertwining maps enter resolvent estimates. The present matrix-domain similarity is isometric, so no such distortion occurs.

Theorem 4.4. For every $\varepsilon > 0$, $\sigma_\varepsilon(B) = \sigma_\varepsilon(C)$. Equivalently, the resolvent norm functions $\lambda \mapsto \|R(\lambda, B)\|$ and $\lambda \mapsto \|R(\lambda, C)\|$ agree pointwise on the common resolvent set.

Proof. The resolvent identity $R(\lambda, B) = U_{A^{-1}}R(\lambda, C)U_A$ and the fact that U_A is an onto isometry imply equality of operator norms. The definition of the pseudospectrum then gives the result.

The theorem is valuable when the transformed matrix is non-normal. In that setting the spectrum alone may underestimate sensitivity, whereas pseudospectra reveal large resolvent regions. Matrix-domain coordinates do not change this sensitivity when the natural graph norm is used; they only change its coordinate representation.

4.4. Riesz projections and isolated spectral components

Similarity also preserves the local spectral data associated with isolated components. Let Γ be a positively oriented contour lying in the resolvent set and enclosing an isolated part Σ of $\sigma(C)$. The Riesz projection is

$$P_C = \frac{1}{2\pi i}\int_\Gamma R(\lambda, C)\, d\lambda. \tag{4.2}$$

Using $R(\lambda, B) = U_{A^{-1}}R(\lambda, C)U_A$ and integrating in operator norm gives $P_B = U_{A^{-1}}P_CU_A$. Hence the spectral subspaces $R(P_B)$ and $R(P_C)$ are isometrically isomorphic. In particular, finite algebraic multiplicities of isolated eigenvalues are preserved.

Theorem 4.5. If λ_0 is an isolated spectral point of B, then it is an isolated spectral point of C with the same Riesz projection rank. If that rank is finite, λ_0 has the same algebraic multiplicity for B and C. The nilpotent parts of $B - \lambda_0 I$ and $C - \lambda_0 I$ on their respective spectral subspaces are similar.

This result strengthens the equality of point spectra. It shows that Jordan-type information carried by generalized eigenspaces is unchanged by the matrix-domain transform. Consequently, residues of the resolvent at poles and finite-dimensional spectral decompositions may be computed on X and transported back without loss.

5. Fredholm properties and essential spectrum

For $\lambda \in \mathbb{C}$, the operator $B - \lambda I$ is Fredholm if it has closed range and finite-dimensional kernel and cokernel. The index is $\mathrm{ind}(B - \lambda I) =$

$\dim\ker(B-\lambda I)-\operatorname{codim} R(B-\lambda I)$. Because U_A is an isomorphism, these quantities are similarity invariants.

Theorem 5.1. $B-\lambda I$ is Fredholm if and only if $C-\lambda I$ is Fredholm. In that case $\operatorname{ind}(B-\lambda I)=\operatorname{ind}(C-\lambda I)$). Consequently, the Fredholm essential spectrum $\sigma_e(T)=\{\lambda: T-\lambda I$ is not Fredholm$\}$ satisfies $\sigma_e(B)=\sigma_e(C)$.

Proof. The kernel dimensions agree because U_A is a bijection between the kernels. The ranges correspond by $R(C-\lambda I)=U_A R(B-\lambda I)$, so closedness and codimension are preserved. Hence Fredholmness and the index coincide.

This observation is useful when the conjugate operator has the form "simple operator + compact perturbation." If $C=C_0+K$ with K compact on X, then $B=VC_0A+VKA$, and VKA is compact on X_A. Standard essential-spectrum results on X can therefore be transferred to X_A. In particular, compact perturbations do not alter the Fredholm essential spectrum, so $\sigma_e(B)=\sigma_e(VC_0A)$.

5.1. Compact perturbations and a Fredholm alternative

Suppose the model operator has the form $C=C_0+K$, where K is compact. For $\lambda\in\rho(C_0)$, factor

$$C-\lambda I=(I+K(C_0-\lambda I)^{-1})(C_0-\lambda I). \quad (5.1)$$

The first factor is identity plus compact. Therefore $C-\lambda I$ is Fredholm of index zero. Transporting through U_A gives the same statement for $B-\lambda I$ on X_A. This yields a useful Fredholm alternative away from the spectrum of the reference operator C_0.

Theorem 5.2. Let $B=U_{A^{-1}}(C_0+K)U_A$ with K compact on X and let $\lambda\in\rho(C_0)$. Then $B-\lambda I$ is Fredholm of index zero. Exactly one of the following occurs: (i) $B-\lambda I$ is bijective; or (ii) $\ker(B-\lambda I)$ is nontrivial and finite-dimensional, the range has the same finite codimension, and the inhomogeneous equation is solvable precisely for right-hand sides lying in that closed range.

In dual form, f belongs to $R(B-\lambda I)$ if and only if every functional in $\ker((B-\lambda I)^*)$ annihilates f. Under the canonical dual isomorphism induced by U_A, this compatibility condition is exactly the corresponding one for $C-\lambda I$. Thus orthogonality or annihilator conditions needed in applications may be verified on the model space.

The result is especially effective when C_0 is diagonal or shift-like and K has finite rank. Then the continuous part of the spectrum is controlled by C_0, while isolated eigenvalues introduced by K can be determined from a finite-dimensional characteristic equation.

5.2. Essential spectrum and compactly generated domain operators

Because compactness is invariant under conjugation by U_A, the Calkin-algebra classes of B and C correspond. Consequently every essential spectrum defined through Fredholm failure is identical for B and C. If $C=C_0+K$ with K compact, then $\sigma_e(B)=\sigma_e(C)=\sigma_e(C_0)$.

This provides a systematic construction principle. Start with a model operator C_0 whose essential spectrum is known, choose a compact K to create desired discrete eigenvalues, and set $B=A^{-1}(C_0+K)A$. The resulting matrix on X_A may be dense or have complicated coefficient dependence, but its essential spectral structure is inherited from C_0 and its discrete spectral changes are those of the compact perturbation.

5.3. Weyl-type stability under compact perturbations

Let C_1 and C_2 be bounded operators on X with C_2-C_1 compact, and let $B_j=U_{A^{-1}}C_jU_A$. Then B_2-B_1 is compact on X_A. Hence the Fredholm essential spectra of C_1, C_2, B_1 and B_2 coincide, and the Fredholm index is unchanged wherever the operators are Fredholm. On a connected component of the complement of the essential spectrum on which the reference operator has index zero and which contains a resolvent point, the analytic Fredholm alternative implies that spectral points introduced by the compact perturbation are isolated eigenvalues of finite algebraic multiplicity; their only possible accumulation is at the boundary of that component, hence at the essential spectrum.

For applications, this means that one can split a

complicated associated matrix into a tractable principal part C_0 and a compact remainder K. The principal part determines the essential spectral backbone, while finite-dimensional or compact corrections determine discrete eigenvalues. Such a decomposition is often more informative than attempting to invert the full infinite matrix directly.

If C_0 is diagonal with $d_n \to d$ and K compact, then $\sigma_e(B) = \{d\}$. If C_0 is the unilateral shift plus a compact operator, then the essential spectrum is the unit circle in settings where the shift has that Fredholm essential spectrum. The precise choice of essential spectrum should always be stated, but the transfer principle itself is independent of that choice whenever it is similarity invariant.

6. Model examples

6.1. Diagonal model

Let $X = \ell_p$, $1 \le p < \infty$, or $X = c_0$, and let $d = (d_n)$ be bounded. Define the diagonal operator M_d on X by $(M_d y)_n = d_n y_n$. Its spectrum is the closure of the set $\{d_n : n \ge 0\}$. Now let A be any triangle and define $B = VM_dA$ on X_A. The operator B may be a complicated infinite matrix in the original coordinates, but $U_A B U_{A^{-1}} = M_d$.

Proposition 6.1. For $B = VM_dA$ on X_A, $\sigma(B) = closure(\{d_n : n \ge 0\})$. If $d_k = \lambda$ for some k, then $Ve^{(k)}$ is an eigenvector of B with eigenvalue λ. For λ outside the closure of $\{d_n\}$, the solution of $(B - \lambda I)x = f$ is

$$x = Vz, \qquad z_n = \frac{(Af)_n}{d_n - \lambda}. \tag{6.1}$$

Formula (6.1) shows the practical value of the transform. A potentially dense-looking matrix equation on X_A is reduced to scalar division in the model coordinates. The only restriction is the same one as for M_d: the denominator must stay uniformly away from zero for bounded invertibility.

6.2. Shift model

Let S be the unilateral right shift on $X = \ell_p$ or c_0, $(Sy)_0 = 0$ and $(Sy)_n = y_{n-1}$. For $q \in \mathbb{C}$, set $C = qS$ and $B = V(qS)A$. The spectrum of S on these spaces is the closed unit disk, hence $\sigma(B) = \{\lambda : |\lambda| \le |q|\}$. When $|\lambda| > |q|$, the resolvent follows from the geometric series

$$(B - \lambda I)^{-1} = -\lambda^{-1} V[\textstyle\sum_{n=0}^{\infty} (q/\lambda)^n S^n]A. \tag{6.2}$$

If A is itself a difference triangle, B is a conjugated shift adapted to the difference coordinates. Such operators are useful test cases because their spectra are exactly known but their original matrices can have several nonzero diagonals and long-range coefficients after multiplication by V and A.

6.3. Projection and finite-rank models

Let P_N be the coordinate projection on X onto $\text{span}\{e^{(0)}, \dots, e^{(N)}\}$ and set $B_N = VP_NA$. Then B_N is a finite-rank projection on X_A. Its spectrum is contained in $\{0,1\}$, and equals $\{0,1\}$ whenever the domain is infinite dimensional and P_N is nonzero. The family B_N converges strongly, but not in operator norm, to the identity on X_A whenever X has AK. This gives a natural system of finite-dimensional approximations inherited from the model space.

6.4. A shift transported to a generalized difference domain

Let $X = \ell_p$, $1 \le p < \infty$, or $X = c_0$, and let $A = I - \theta S$ be a first-order generalized difference triangle with arbitrary θ for which the matrix domain is defined with its natural norm. Define $B = A^{-1}SA$ on X_A. Then $U_A B U_{A^{-1}} = S$. Hence B has exactly the same spectrum, resolvent norms and spectral subdivisions as the unilateral right shift on X.

In this particular example the conjugation is actually trivial. Because $A = I - \theta S$ is a polynomial in S, A commutes with S and so $A^{-1}SA = S$. Thus $B = S$ as an operator on the matrix domain. The example is therefore best regarded as a consistency check for the transfer principle, not as an example of a complicated transported shift. A genuinely nontrivial transported shift requires an invertible triangle that does not commute with S, for example a variable-coefficient triangle.

For $|\lambda| > 1$ the resolvent has the Neumann

expansion $R(\lambda, S) = -\sum_{n\geq 0} \lambda^{-n-1} S^n$, and therefore

$$R(\lambda, B) = -A^{-1}\left[\sum_{n=0}^{\infty} \lambda^{-n-1} S^n\right] A, |\lambda| > 1. \quad (6.3)$$

The series converges in operator norm on the model space and, after conjugation, in $B(X_A)$. Thus an explicit solution of $(B - \lambda I)x = f$ is obtained by first transforming $g = Af$, applying the shift resolvent series, and reconstructing $x = A^{-1}y$.

6.5. Diagonal models and closure of the diagonal set

Let $d = (d_n)$ be bounded and let M_d be the diagonal multiplier $(M_d y)_n = d_n y_n$ on c_0 or ℓ_p. Define $B = A^{-1}M_d A$. Then $\sigma(B) = \sigma(M_d) = cl\{d_n : n \geq 0\}$. If λ lies outside this closure, the solution is coordinatewise

$$x = A^{-1}y, \qquad y_n = \frac{(Af)_n}{d_n - \lambda}. \quad (6.4)$$

and the resolvent norm is $\sup_n |d_n - \lambda|^{-1}$. This furnishes a large class of matrix-domain operators with completely explicit solvability. The eigenvalues depend on the model space and on repetitions of the diagonal values, but the full spectrum is always the closure of the diagonal set in these classical sequence spaces.

A useful variant is an asymptotically constant diagonal $d_n \to d$. Then $M_d - dI$ is compact on c_0 and ℓ_p because the diagonal coefficients tend to zero. Hence $B - dI$ is compactly generated through the domain transform, $\sigma_e(B) = \{d\}$, and all spectral points away from d are isolated diagonal values of finite multiplicity when those values occur only finitely often.

6.6. Rank-one perturbations and a scalar solvability condition

Let $C = C_0 + u \otimes \varphi$, where $(u \otimes \varphi)y = \varphi(y)u$ is rank one and $\lambda \in \rho(C_0)$. Put $R_0 = R(\lambda, C_0)$. Solving $(C - \lambda I)y = g$ and applying R_0 gives $y = R_0 g - R_0 u\, \varphi(y)$. Applying φ yields the scalar equation

$$[1 + \varphi(R_0 u)]\varphi(y) = \varphi(R_0 g). \quad (6.5)$$

If $1 + \varphi(R_0 u) \neq 0$, the solution is unique and can be substituted back into the preceding formula. If the scalar factor vanishes, λ becomes an eigenvalue of the rank-one perturbation provided $R_0 u \neq 0$. For $B = A^{-1}CA$ on X_A the same criterion applies verbatim, with u, φ and g expressed in model coordinates. This example shows how an infinite-dimensional domain equation can reduce to a one-dimensional compatibility condition after conjugation.

6.7. A transported weighted shift model

Let W be a unilateral weighted shift on ℓp or c0, $(Wy)_0 = 0$ and $(Wy)_n = w_{n-1}y_{n-1}$, with bounded weight sequence w. Define $B = A^{-1}WA$ on X_A. Then all spectral data of B coincide with those of W. Even when an explicit closed form for $\sigma(W)$ is unavailable, standard weighted-shift estimates immediately transfer.

For example, $\|W^n\| = \sup_k |w_k w_{k+1} \cdots w_{k+n-1}|$. Therefore,

$$r(B) = r(W) = \lim_{n\to\infty}\left(\sup_k |w_k \cdots w_{k+n-1}|\right)^{1/n}. \quad (6.6)$$

where the limit may equivalently be written as an infimum of the n[th]-root norms by submultiplicativity. This gives a computable spectral-radius formula for a potentially dense matrix B on X_A.

If the weights converge to $w \neq 0$, then $W - wS$ is compact. Hence B is a compact perturbation of $A^{-1}(wS)A$, and its Fredholm essential spectrum is inherited from the constant-weight shift. On a connected component of the complement on which the Fredholm index is zero and which contains a resolvent point, analytic Fredholm theory implies that any additional spectral points are isolated eigenvalues of finite algebraic multiplicity.

7. Perturbation and stability of solvability

Similarity also converts perturbation questions into standard model-space estimates. Suppose $B = B_0 + E$ on X_A and $C_j = U_A B_j U_{A^{-1}}$, $F = U_A E U_{A^{-1}}$. Then $\|F\| = \|E\|$. If $\lambda \in \rho(B_0)$

and $\|E(B_0 - \lambda I)^{-1}\| < 1$, the perturbed equation remains uniquely solvable.

Theorem 7.1. Let $\lambda \in \rho(B_0)$. If $\|E\| \cdot \|(B_0 - \lambda I)^{-1}\| < 1$, then $\lambda \in \rho(B_0 + E)$ and

$$(B_0 + E - \lambda I)^{-1} =$$

$$(B_0 - \lambda I)^{-1}[I + E(B_0 - \lambda I)^{-1}]^{-1}. \quad (7.1)$$

Proof. Factor $B_0 + E - \lambda I =$

$[I + E(B_0 - \lambda I)^{-1}](B_0 - \lambda I)$. The bracket is invertible by a Neumann series. The same factorization holds for the conjugate operators on X, with identical norms under U_A.

This criterion is especially useful when C_0 has a diagonal or shift form and F is represented by a small matrix perturbation. Instead of estimating the original perturbation relative to the matrix-domain coordinates, one may work entirely in X and then transport the conclusion back to X_A.

7.1. Resolvent growth

Because the similarity is isometric, the pseudospectral behavior is also preserved in the norm induced by A. For $\varepsilon > 0$, define the ε −pseudospectrum by $\sigma_\varepsilon(T) = \{\lambda: \lambda \in \sigma(T)$ or $\|(T - \lambda I)^{-1}\| > \varepsilon^{-1}\}$. Equation (3.3) gives $\sigma_\varepsilon(B) = \sigma_\varepsilon(C)$. Hence the sensitivity of the resolvent to perturbations is not worsened by passing to the matrix domain when the natural norm is used. This is a stronger statement than ordinary spectral invariance and is useful for numerical truncations of infinite matrices.

7.2. Resolvent perturbation bounds and spectral exclusion regions

Let C be a bounded model operator and E another bounded operator. If $\lambda \in \rho(C)$ and $\|E\, R(\lambda, C)\| < 1$, then $C + E - \lambda I = (I + ER(\lambda, C))(C - \lambda I)$ is invertible. Therefore

$$\| E \| \| R(\lambda, C) \| < 1 \quad \Rightarrow \quad \lambda \in \rho(C + E). \quad (7.2)$$

Under the domain transform, if $B = A^{-1}CA$ and $F = A^{-1}EA$, then $\|F\| = \|E\|$ and the same exclusion criterion holds for $B + F$. Consequently perturbation radii can be computed on X without introducing norm-equivalence constants.

The criterion also gives the standard perturbation inclusion: for every $\varepsilon > \|E\|$, $\sigma(C + E) \subset \sigma_\varepsilon(C)$. Since pseudospectra are exactly preserved by the isometric matrix-domain similarity, for every $\varepsilon > \|F\|$ one likewise has $\sigma(B + F) \subset \sigma_\varepsilon(B)$. This formulation avoids the boundary ambiguity that arises when the pseudospectrum is defined using the strict inequality $\|R(\lambda, T)\| > \varepsilon^{-1}$.

7.3. Finite sections: use and limitation

Infinite matrix problems are often approximated by truncating to the first N coordinates. Conjugation suggests that finite sections should be formed after, or consistently with, the matrix-domain transform. If P_N is the coordinate projection on X, the model finite section is $P_N C P_N$. A domain-adapted approximation is then $A^{-1}P_N C P_N A$ on the range where the products are meaningful.

Finite-section eigenvalues need not converge to the infinite-dimensional spectrum for arbitrary non-normal operators, so the method should not be used as a spectral theorem without additional stability hypotheses. Nevertheless, resolvent estimates offer a rigorous validation device: an approximate eigenvalue λ_N is reliable only when accompanied by residual vectors whose transformed defects are small and by a stability analysis that rules out pseudospectral pollution. The isometric transform ensures that residual norms computed in X are exactly the residual norms in X_A.

7.4. Analytic dependence on a scalar parameter

Suppose $C(t)$ is an operator-valued analytic function of a complex parameter t in a domain Ω and define $B(t) = U_{A^{-1}} C(t) U_A$. Then $B(t)$ is analytic in $B(X_A)$ and all resolvent identities transfer uniformly on compact parameter sets. If λ_0 is a simple isolated eigenvalue of $C(t_0)$, standard analytic perturbation theory supplies a local analytic eigenvalue branch $\lambda(t)$ and a corresponding Riesz projection. The same branch and projection rank describe $B(t)$.

This observation is useful when coefficients of an infinite matrix depend on a physical or numerical parameter. Rather than differentiating the complicated domain matrix, one may differentiate $C(t)$. For a simple eigenvalue with normalized right/left eigenvectors in a Hilbert model, the first derivative can be computed from the model perturbation, and the resulting eigenvalue derivative is automatically the derivative for the matrix-domain operator.

Even without differentiability of individual matrix entries in the original coordinate representation, analyticity of the bounded model family is enough. This highlights again that operator topology, not formal entrywise manipulation, is the appropriate level for spectral perturbation.

7.5. Resolvent identity and comparison of two operators

For λ in the common resolvent set of C and $C + E$, the second resolvent identity gives

$$R(\lambda, C+E) - R(\lambda, C) = -R(\lambda, C+E)ER(\lambda, C). \quad (7.3)$$

Conjugation yields exactly the same identity for B and $B + F$, where $F = U_{A^{-1}}EU_A$, with equal norms of every factor. Therefore,

$$\| R(\lambda, B+F) - R(\lambda, B) \| \leq \| R(\lambda, B+F) \| \| F \| \| R(\lambda, B) \|. \quad (7.4)$$

No condition number of A appears because U_A is an isometry between the graph-norm domain and the model space. This is a significant practical advantage over working with the same matrices under an unrelated ambient norm, where conjugation could amplify perturbations by $\|A\|\|A^{-1}\|$.

The estimate can be iterated to control resolvent changes under a sequence of small perturbations. In numerical approximation, it also provides a consistency test: if a finite or truncated model C_N approximates C in operator norm on an invariant subspace, then resolvent convergence away from the spectrum transfers to the corresponding domain operators without additional constants.

7.6. Spectral interpretation of the domain norm

The natural norm $\|x\|_A = \|Ax\|_X$ is not merely a convenient completeness device. It is the unique choice in this framework that makes the coordinate transform U_A an isometry and therefore makes all resolvent norms, pseudospectra and minimum moduli exactly invariant. If instead X_A is equipped with an equivalent but non-isometric norm, the spectrum remains invariant under bounded similarity but quantitative data acquire condition-number factors.

Specifically, if $J: X_A \to X$ is a bounded isomorphism with constants $\|J\|$ and $\|J^{-1}\|$, then $\|R(\lambda, B)\|$ can differ from $\|R(\lambda, JBJ^{-1})\|$ by at most the factor $\kappa = \|J\|\|J^{-1}\|$ in either direction. Pseudospectral levels are correspondingly distorted. The graph norm eliminates this artifact, which is why it is the correct norm for the exact transfer theory developed here.

7.7. Spectral decomposition for separated components

Suppose $\sigma(C) = \Sigma_1 \cup \Sigma_2$ where Σ_1 and Σ_2 are disjoint compact subsets separated by a simple closed contour Γ in the resolvent set. The Riesz projection P_C associated with Σ_1 commutes with C and yields the invariant decomposition $X = R(P_C) \oplus N(P_C)$. Since $P_B = U_{A^{-1}}P_C U_A$, the matrix-domain space decomposes as $X_A = R(P_B) \oplus N(P_B)$, and B reduces to the corresponding restrictions on these two invariant subspaces.

The spectra of the restrictions are Σ_1 and Σ_2, respectively. Hence a complicated infinite matrix on X_A can be split into spectral blocks whenever the model operator admits such a separation. This is especially useful when one component consists of finitely many isolated eigenvalues and the other contains the essential spectrum. The finite-dimensional spectral subspace can then be analyzed by ordinary linear algebra after transformation.

If $\Sigma_1 = \{\lambda_1, \dots, \lambda_s\}$ consists of isolated poles of the resolvent with finite algebraic multiplicity, then $R(P_B)$ has dimension equal to the sum of those multiplicities. On that subspace, B is

similar to a finite-dimensional Jordan representation of the corresponding part of C. The complementary restriction contains the remaining spectrum and is unaffected by the finite-dimensional reduction.

7.8. Solvability under a bounded forcing operator

Many applications involve equations of the form $(B-\lambda I)x = Gz$, where z belongs to another Banach space Z and $G: Z \to X_A$ is bounded. Set $\hat{G} = U_A G: Z \to X$. If $\lambda \in \rho(B)$, then the solution operator is

$$x = (B-\lambda I)^{-1} Gz = U_{A^{-1}} R(\lambda, C)\hat{G}z. \qquad (7.5)$$

Therefore, $\|(B-\lambda I)^{-1}G\| = \|R(\lambda, C)\hat{G}\|$. This formulation separates the internal spectral difficulty, represented by $R(\lambda, C)$, from the way external data enter the system, represented by $\hat{G}$. If G itself is defined by an infinite matrix, its transformed form is obtained by left multiplication with A.

The same idea treats observation operators. If $H: X_A \to Y$ is bounded, then the input-output transfer operator $H(B-\lambda I)^{-1}G$ equals $(HU_{A^{-1}})R(\lambda, C)(U_A G)$. Thus frequency-domain or resolvent estimates for a matrix-domain system reduce to an ordinary model-space sandwich. Compactness of either transformed input or output factor can then be combined with the spectral information developed above.

7.9. A criterion for uniqueness at spectral boundary points

At $\lambda \in \sigma(B)$, failure of invertibility can arise from nontrivial kernel, non-dense range, non-closed range or unbounded inverse behavior. Similarity distinguishes none of these mechanisms; they are transferred exactly, but it makes them easier to identify when C is standard. In particular, if $\lambda \in \sigma(C)$ but $C-\lambda I$ is injective, then the homogeneous equation on X_A has only the zero solution even though the inhomogeneous equation cannot be solved boundedly for all data.

This situation occurs frequently for unilateral shifts and other non-normal operators. It cautions against equating "spectral value" with "eigenvalue." For matrix-domain problems, direct recursions may misleadingly suggest uniqueness and therefore invertibility. The correct conclusion is that uniqueness addresses only the kernel; surjectivity and boundedness of the inverse must also be checked on the model space.

A useful sufficient condition for uniqueness is bounded below: if $m(C-\lambda I) > 0$, then $\ker(C-\lambda I) = \{0\}$ and $R(C-\lambda I)$ is closed. Through the isometry, the same minimum modulus holds for $B-\lambda I$. If, in addition, the range is dense, then it is all of X_A and λ lies in the resolvent. Thus the three pieces: kernel, closed range and density, can be verified without ever expanding the domain norm coordinate by coordinate.

7.10. Practical workflow for an infinite matrix on X_A

The results can be organized into a short workflow for concrete problems. First verify that A is an invertible triangle and work with the natural norm on X_A. Second compute, either exactly or structurally, the conjugated operator $C = ABA^{-1}$. Third establish boundedness of C on the model space before making spectral claims. Only after this step should one use diagonal, shift, compact-perturbation or other model theorems to determine $\sigma(C)$ and its subdivisions.

For a fixed λ, solvability is then decided by $C-\lambda I$. If it is invertible, the solution is $x = A^{-1}(C-\lambda I)^{-1}Af$. If it is merely injective, uniqueness holds but existence for arbitrary f may fail. If it is Fredholm, the finite-dimensional kernel and cokernel give the precise compatibility conditions. This order of analysis prevents formal coordinate recursion from being mistaken for bounded invertibility.

Finally, quantitative questions should be asked in the same coordinates. Resolvent norms, minimum moduli, pseudospectra and perturbation thresholds are exactly preserved by the isometry. Thus there is no advantage in estimating them directly in the original matrix-domain coordinates unless the conjugated matrix itself is unavailable. The model-space representation is not an approximation; it is an

exact reformulation of the operator problem.

This workflow applies equally to difference domains, generalized weighted means and products of triangles. The only technically delicate step is often the justified computation of the infinite matrix product ABA^{-1}; when formal multiplication is problematic, the conjugated operator can still be defined abstractly by $U_A B U_{A^{-1}}$ and studied through bounded-operator arguments.

8. Concluding remarks

The matrix domain X_A is best viewed as the model space X written in coordinates adapted to the triangle A. Once this viewpoint is adopted, the spectral theory of bounded operators on X_A becomes a similarity problem. The transformation $C = ABA^{-1}$, interpreted operator-theoretically as $U_A B U_{A^{-1}}$, transfers solvability, the resolvent, the spectrum, fine spectral subdivisions, Fredholm properties and the essential spectrum to X. Under the natural norm, even operator and resolvent norms are preserved exactly.

The method does not replace direct spectral analysis when the conjugated operator C is itself complicated, but it gives a principled first reduction. In many applications A is chosen precisely because it simplifies a difference, summability or recurrence structure; consequently the conjugate matrix can be more transparent than B. Diagonal and shift models show that highly nontrivial-looking operators on a matrix domain may have immediately computable spectra.

The same change of coordinates also transports compactness. For an operator from X_A into another sequence space, the associated matrix BA^{-1} acts on X; for endomorphisms of X_A, compactness is equivalent to compactness of ABA^{-1}. Combining this with the Hausdorff measure of non-compactness gives a quantitative route to compactness criteria on matrix domains.

9. Further extensions and computational implications

The similarity framework extends immediately to closed densely defined operators whenever the domain is transported correctly. If an unbounded operator B on X_A has domain $\mathrm{Dom}(B)$, then $C = U_A B U_{A^{-1}}$ has domain $U_A \mathrm{Dom}(B)$. Closedness, density of the domain, kernels and ranges are preserved by U_A, so much of the resolvent theory continues to hold. This observation opens the matrix-domain method to discrete differential operators whose coefficients grow and therefore do not define bounded maps on the whole space.

Finite-section computation also benefits from the transformed coordinates. Direct truncation of B on X_A can be misleading because the norm is not the ordinary coordinate norm. A more natural procedure is to truncate C on X using the canonical projections and then return to X_A through V and A. When X has AK, the finite-rank operators $VP_N C P_N A$ approximate the model calculation in coordinates compatible with the domain norm. Such approximations do not automatically converge in operator norm for every bounded C, but they provide a consistent basis for Galerkin-type and finite-section schemes.

The pseudospectral identity obtained in Section 7 is particularly relevant to stability. It shows that any large resolvent norm observed for B is already present for C and is not an artifact of the matrix-domain coordinate change. Hence spectral sensitivity can be studied where the operator has the simplest representation. For difference domains this often means analyzing an operator on ℓ_p or c_0 rather than working with nested difference recurrences.

A useful research program is therefore to classify families of matrices B for which ABA^{-1} has a recognized structure: diagonal, weighted shift, Toeplitz, banded, compact-plus-diagonal, or a small perturbation of one of these classes. Such a classification would convert many domain-specific spectral calculations into instances of standard operator models and would make the dependence on the defining triangle A explicit rather than hidden in coordinate manipulations.

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